D.K.M. COLLEGE FOR WOMEN (AUTONOMOUS),VELLORE.

ABSTRACT ALGEBRA

II B.Sc. MATHEMATICS

Unit – I SECTION-A

1. If G is a group, prove that the identity element of a group is unique.
2. Define Subgroup.
3. State Euler's theorem.
4. Define the order of an element of a group.
5. Show that if every element of a group G is its own inverse, then G is abelian.
6. Let G be a group. Let a,b $\in$ G. Then prove that $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$.
7. Prove that the identity element of a group is unique.
8. Define cyclic group and give an example of a cyclic group.
9. Define Right and Left Cost.
10. Define Abelian group.
11. Define order of a group
12. Define order of an element.

SECTION – B

13. If G is a group prove that $(a.b)^2=a^2.b^2$ if and only if G is an abelian group.

14. If H and K are subgroup of group G then prove that $H \cap K$ is also a subgroup of G.

15. Prove that the union of two subgroup need not be a subgroup.

16. If G is a finite group and a$\in$G, Prove that O(a)/O(G).

17. Prove that the relation $a \equiv b \bmod H$ is an equivalence relation.

18. State and Prove Euler's Theorem.

19. State and Prove Fermat's Theorem.

20.Prove that a non empty subset H of a group G is a subgroup of G if and only if

i)a,b $\in$ H$\Rightarrow$ ab $\in$ H ii) a $\in$ H $\Rightarrow$ a^{-1} $\in$ H.

SECTION – C

21.Prove the necessary and sufficient condition for a non empty subset H to the subgroup is that a,b$\in$ H$\Rightarrow$ ab^{-1} $\in$ H .

22.Prove that the union of two subgroups of a group G is a subgroup if and only if one is contained in the other.

23.State and prove Lagrange's theorem.

24.If G is a finite group and H is a subgroups of G, then O(H) is a divisor of O(G).

25.If H is a nonempty finite subset of a group G and H is closed under multiplication, then H is a subgroup of G.

UNIT- II SECTION-A

1. Prove that the intersection of two normal subgroups of a group G is a normal sub group of G.
2. Define Kernel of a group homomorphism.
3. If φ Is homomorphism of G into $\bar{G}$, then prove that $\varphi(e) = \bar{e}$.
4. Define Normal subgroup.
5. Prove that every subgroup of an abelian group is a normal subgroup.
6. Define homomorphism of a group and give an example.
7. Prove the intersection of two normal subgroup of a group G is a normal subgroup of G.
8. Let θ:G→G' be a homomorphism of G onto G'. If G is abelian, then prove that G' is also abelian.
9. If θ: G→G' is a homomorphism of group. Show that θ(e)=e' and $(\theta(a))^{-1} = \theta(a^{-1})$ for all a $\in$G.
10. Define Isomorphism with example.

11.Define Automorphism with example.

SECTION – B

13.Prove that if N is the normal subgroup of G if and only if gNg^{-1} =N for every g∈G.

14.If N and M are normal subgroups of G, Prove that N∩M is also a normal subgroup of G.

15.If H is a subgroup of G and N is normal subgroup of G then prove that H∩N is normal in H.

16.Prove that subgroup N of G is normal subgroup of G if and only if gNg^{-1} =N for all g∈G.

17.Prove that if a subgroup N of G is a normal subgroup of G if and only if the product of the two right cosets of N in G is again a right coset of N in G.

18.If G is a group and N is a normal subgroup of G, Prove that ${}^{G}/_{N}$ is a group.

19.Prove that a homomorphism φ of G into $\bar{G}$ with kernel k_{φ} is an isomorphism if and only if kernel of φ={e}.

20.Prove that if a subgroup N of G is a Normal subgroup of G ⇔ Every left coset of N in G is a right coset of N in G.

21.Prove that a homomorphism φ of G into $\bar{G}$ with kernel k_{φ} is the normal subgroup of G.

SECTION - C

21.State and prove first isomorphism theorem.

22.Let φ be a homomorphism of G onto $\bar{G}$ with kernel K. Then prove that G/K ≅ $\bar{G}$.

23.If H and K are finite sub groups of G of orders O(H) and O(K). respectively, then

$$O(HK)= O(H)O(K)/ O(H\cap K)$$

24.Prove that HK is a subgroup of G if and only if HK=KH.

UNIT – III SECTION - A

1. Define automorphim.
2. Define even permutation with an example.
3. Prove that for any abelian group G, θ: G→G given by θ(a) = a^{-1} is an automorphism.
4. Define cycle and transposition.
5. Express the following permutation as a product of disjoint cycle. (4 2 1 5) (3 4 2 6) (5 6 7 1).
6. Prove that the inverse of the identity permutation I is an even permutation.
7. Prove that the inverse of the even permutation is even.
8. Express the following permutation as a product of disjoint cycles$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$
9. Express the following permutation as a product of disjoint cycles$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$

SECTION – B

10. Prove that (I(G), ·) is a subgroup of A(G).
11. If G is a group, then prove that A(G), the set of automorphism of G is also a group.
12. Prove that I(G) is isomorphic to G/Z where Z is the center of G.
13. Prove that the set of all even permutations is the normal subgroup of S_n.
14. Prove that every infinite cyclic group is isomorphic to Z.
15. Prove that every finite cyclic group is isomorphic to Z_n.
16. Prove that every permutation is the product of its disjoint cycles.
17. Prove that every permutation is the product of transposition.
18. Prove that the set of all even permutation is a normal subgroup of S_n.
19. Prove that S_n has a normal subgroup of index 2, the alternating group A_n consisting of all even permutations.

20. Prove that the product of

i. Two even permutation is even.

ii. Two odd permutation is even.

iii. An even permutation and an even permutation is odd.

21. Prove that out of $\lfloor n$ permutation $\frac{\lfloor n}{2}$ are odd permutation and $\frac{\lfloor n}{2}$ are even permutation.

SECTION – C

22. State and prove Cayley's Theorem.

23. If G is a group, then prove that A(G), the set of automorphism of G is also a group.

24. Let S be any finite set of n elements then prove that the set of all permutation on S is a group under the product of permutation.

25. Prove that S_n has a normal subgroup of index 2, the alternating group A_n consisting of all even permutations.

UNIT – IV – RINGS **SECTION - A**

1. Define ring and a commutative ring.
2. A sub ring of R need not be an ideal of R. Give an example.
3. Define Integral domain.
4. Define finite characteristic.
5. Define integral domain.
6. Define zero divisors with an example.
7. Define Homomorphism of Rings.
8. Define isomorphism of Rings.
9. Define Kernal of φ.
10. If φ is a homomorphism from R onto R' then prove i) φ(0) = 0 ii) φ(-a) = - φ(a).
11. Prove that every ideal of ring R I a sub ring of R.
12. If R is a ring for all a, b$\in R$

    a.0=0.a=0 (ii) a(-b)=(-a)b=-(ab)
13. Define Ideal of a Ring.
14. Define Quotient Ring.
15. Give an example of a sub ring which need not be an ideal.

16. If U is an ideal of R and $1 \in U$ prove that U=R.

17. If F is a field then prove its only ideal are {0} and itself.

SECTION - B

18. Prove that every field is an integral Domain.

19. Prove that any finite integral domain is a field.

20. If φ is a homomorphism of R into R' with kernel $I(\varphi)$, prove that

    (i) I (φ) is a subgroup under addition

    (ii) If a$\in I(\varphi)$ r $\in$R then $ar \in I(\varphi) and\ ra \in I(\varphi)$.

21. Prove that a homomorphism of R$\rightarrow R'$ is an isomorphism $\Leftrightarrow$ $I(\varphi)$ = {0}.

22. If U and V are ideals of R then prove that U +V = {u+v/u$\in$ U, v$\in$V} is a ideal.

23. Prove that the intersection of two ideals is an ideal.

SECTION – C

26. If U is an ideal of the ring R then prove that R/U is a ring and R/U is homomorphic image of R.

27. Prove that a finite integral domain is a field.

28. Let R be a commutative ring with unit element whose only ideals are {0} are R itself then R is a field.

29. Prove that first isomorphism theorem for rings.

UNIT – V – RINGS SECTION - A

1. Define Maximal ideal.

2. Define Euclidean Ring.

3. Define Principal ideal ring.

4. Prove that any Euclidean domain of R has an identity element.

5. Let R be a Euclidean ring suppose that a,b,c $\in$R a/bc and (a,b)=1 then prove that a/c.

6. Prove that the element a in a Euclidean ring is a unit $\Leftrightarrow$ d(a) = d(1).

SECTION - B

7. Let R be a Euclidean ring. Then prove that any two elements a and b in R have a greatest common divisor d. Moreover d= λa+μb for some λ, $\mu \in$ R.

8. If π is a prime element in the Euclidean ring and π/ab then prove that either π/a or π/b.
9. Prove that every non element is a Euclidean ring can be uniquely written as a product of prime elements or is unit in R.
10. Let R be a Euclidean ring a,b $\in$ R if b $\neq$ 0 is not a unit in R then Prove that d(a) < d(ab).

SECTION- C

1. If R is a commutative ring with unit element and M is an ideal of R, then prove that M is a maximal ideal of R if and only if R/M is a field.
2. Let R be a Euclidean ring a,b $\in$ R if b $\neq$ 0 is not a unit in R then Prove that d(a) < d(ab).
3. Prove that the ideals A=(a_0) is a maximal ideal of the Euclidean ring R if and only if a_0 is a prime element of R.
4. Let R be a Euclidean ring then prove that every element in R is either a unit in R or can be written as the product of a finite number of prime element of R.
5. State and prove unique factorization theorem.
6. Prove that every non element is a Euclidean ring can be uniquely written as a product of prime elements or is unit in R.