D.K.M COLLEGE FOR WOMEN (AUTONOMOUS), VELLORE

DEPARTMENT OF MATHEMATICS

SUB: ALGEBRA **SUB CODE: 15CPMA1A**

SECTION- A UNIT-1

1. Define conjugate of a in G and prove that conjugacy is an equivalence relation on G
2. Define normalizer of a in G and prove that N(a) is a subgroup of G.
3. If G is a finite group then prove that $C_a = \frac{O(G)}{O(N(a))}$
4. Prove that the number of elements conjugate to a in G is the index of the normalizer of a in G.
5. State and prove class equation.
6. Prove that $O(G) = \sum \frac{O(G)}{O(N(a))}$
7. If a£ Z if and only if N(a)=G. If G is finite, a£Z iff O(N(a))= O(G).
8. If $O(G)=p^n$, p is a prime number then $Z(G) \neq \{e\}$.
9. If $O(G)=p^2$, where p is a prime number then G is abelian.
10. State and prove Cauchy's theorem for finite group.
11. If p is a prime number and p/O(G) then G has an element of order p.
12. If $p^m | O(G)$, $p^{m+1} \nmid O(G)$ then G has a subgroup of order p^m.
13. Prove that $n(k)= 1+p+\ldots+p^{k-1}$.
14. If A and B are finite subgroup of G then $O(AxB) = \frac{O(A).O(B)}{O(A \cap xBx^{-1})}$
15. Define internal and external direct product
16. Suppose that G is the internal direct product of $N_1, N_2, \ldots\ldots\ldots N_n$ then for $i \neq j$, $N_i \cap N_j = \{e\}$ and if a£N_i, b£N_j then ab=b
17. Let G be a group and suppose that G is the internal direct product of $N_1, N_2, \ldots\ldots\ldots N_n$. Let $T = N_1 X N_2 X \ldots\ldots\ldots X N_n$ then G and T are isomorphic.
18. Obtain class equation of S_3.
19. State and prove third part of sylow's theorem.
20. Prove that the number of p- sylow subgroup in G for a given prime is of the form 1+kp

SEC-B

21. State and prove first part of sylow's theorem.
22. State and prove second part of sylow's theorem.
23. State and prove third part of sylow's theorem.

24. State and prove fundamental theorem on finitely generate R-module.
25. If P is a prime number and $p^{\alpha}|O(G)$ then G has a subgroup of order p^{α}

UNIT-II SEC-A

1. Define module and give one example.
2. Define solvable and give one example.
3. G is solvable iff $G^{(k)}=\{e\}$ for some integer K.
4. Prove that homomorphic image of a solvable group is solvable.
5. Prove that subgroup of a solvable group is solvable.
6. If G is solvable group and if $\overline{G}$ is a homomorphic image of G then $\overline{G}$ is solvable.

SEC-B

7. Prove that any finite abelian group is the direct product of cyclic groups.
8. Prove that S_n is not solvable for n≥5.
9. Let G= S_n, n≥5 then $G^{(k)}$, K=1,2….. Contains every 3-cycle of S_n.

UNIT-III SEC-A

1. If $W \subset V$ is invariant under T then T induces a linear transformation $\overline{T}$ on $\frac{V}{W}$ defined by (V+W) $\overline{T}$ =VT+W. if satisfies q(x)£F[x] then so does $\overline{T}$. If P1(x) is the minimal polynomial for $\overline{T}$ over F and if P(x) is that for T1 then $P_1(x)/P_2(x)$.
2. If V is n-dimensional over F and if T£A(V) has all its characteristic roots in F then T satisfies a polynomial of degree n over F.
3. If $V = V_1 \oplus V_2 \oplus .... \oplus V_k$, where each subspace V_i is of dimension n_i and is invariant under T, an element A(V) then a basis ofV can be found so that the matrix of T in this is of the form $\begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_k \end{pmatrix}$ where each A_i is an n_ixn_i matrix of the liner transformation induced by T on V_i.
4. If T£A(V) is nilpotent then $\alpha_0+\alpha_1T+....\alpha_mT^m$ where α_i£F is invertible if $\alpha_0 \neq 0$.
5. If u£V_1 is such that $uT^{n_{1-k}}$ =0 where 0<k<n, then $u=u_0T^k$ for some u_0£V_1.
6. If M of dimension m is cyclic with respect to T then the dimension of MT^K is m-k for all k<m.
7. Suppose that $V = V_1 \oplus V_2$ where V_1 and V_2 are subspaces of V invariant under T. Let T_1 and T_2 be the linear transformations induced by T on V_1 and V_2 respectively. If the minimal polynomial of T_1 over F is $P_1(x)$ while that of T_2 is $P_2(x)$ then the minimal polynomial for T over F is the LCM of $P_1(x)$ and $P_2(X)$

SEC-B

1. Let T£A(V) has all its characteristics roots in F then the is a basis of V in which the matrix of T is triangular.
2. If A£F_n has all its characteristics roots in F then there is a matrix C£F_n such that CAC^{-1} is triangular.
3. If T£A(V) is nilpotent of index of nilpotence n_1 then a basis of V can be found such that the matrix of T in the basis has the form $\begin{pmatrix} M_{n_1} & 0 & 0 \\ 0 & M_{n_2} & 0 \\ 0 & 0 & M_{n_r} \end{pmatrix}$ where $n_1 \geq n_2 \geq ..... \geq n_r$ and $n_1 + n_2 + ...n_r = \dim V$.
4. Prove that there exists a subspace W of V is invariant under T such that $V = V_1 \oplus W$.
5. Prove that the two nilpotent liner transformation are similar iff and only if they have the same invariant.

UNIT-IV SEC-A

1. If $V = V_1 \oplus V_2 \oplus ..... \oplus V_k$ where each V_i is invariant under T and if $P_i(x)$ is the minimal polynomial over F of T_i, the liner transformation induced by T on V_i, then the minimal polynomial of T over F is the L.C.M. of $p_1(x), p_2(x), ...p_k(x)$.
2. State and prove Jordan theorem
3. Prove that for each i= 1,2,……k, $V_i \neq 0$ and $V = V_1 \oplus V_2 \oplus ..... \oplus V_k$ the minimal polynomial of T_i is $q_i(x)$
4. If all the distinct characteristic roots $\lambda_i .....\lambda_k$ of T lie in F then $V = V_1 \oplus V_2 \oplus ..... \oplus V_k$ where $V_i = \{v \varepsilon V \setminus v(T - \lambda_i)^{l_i} = 0\}$ and where T_i has only one characteristic root λ_i on V_i.
5. Let T£A(V) has all its characteristic roots $\lambda_1 ......\lambda_n$ in F then a basis of V can be found in which the matrix T is of the form $\begin{pmatrix} J_1 & & \\ & J_2 & \\ & & J_k \end{pmatrix}$ where each $J_i = \begin{pmatrix} B_{i_{11}} & & \\ & B_{i_2} & \\ & & B_{i_{ri}} \end{pmatrix}$ and where $B_{i_{11}} B_{i_2} ....B_{i_{ri}}$ are basic Jordon blocks belonging to λ_i
6. Suppose that T in A(V) has minimal polynomial over F the polynomial $P(x) = \gamma_0 + \gamma_1 x + ... + \gamma_{r=1} x^{r-1} + x^r$ Then there is a basis of V over F,

$$m(T) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma_0 & -\gamma_1 & ..... & -\gamma_{r-1} \end{pmatrix}$$

Sec-B

1. State and Prove Jordon theorem?
2. If $P(x)=q(x)^e$ then $m(T)=\begin{pmatrix} C\left(q(x)^{e_1}\right) & & \\ & & \\ & & C\left(q(x)^{e_r}\right) \end{pmatrix}$
3. If T£A(V) has $P(x)=q_1(x)^{t_1}.....q_k(x)^{t_k}$ then $m(T)=\begin{pmatrix} R_1 & & \\ & R_2 & \\ & & R_k \end{pmatrix}$ where

   $R_i = \begin{pmatrix} C\left(q_i(x)^{e_{1i}}\right) & & \\ & & \\ & & C\left(q_i(x)^{e_{iri}}\right) \end{pmatrix}$
4. Let V and W be two vector spaces over F and suppose that ψ is a isomorphism of V onto W suppose S£A_F(V), T£A_F(W) such that $(VS)\psi=(V\psi)T$ Then S and T have the same elements divisors.

Unit-V SEC-A

1. For A,B£F_n and $\lambda\varepsilon F$, i) $tr(\lambda A)=\lambda trA$ ii) tr(A+B)=trA+trB iii)tr(AB)=tr(BA).
2. If T£A(V) then trT is the sum of the characteristic roots of T.
3. If F is a field of characteristic 0 and if T£A(V) is such that trT^i for all $i\geq 1$ then T is nilpotent.
4. If F is of characteristic o and if S,T£A(V) are such that ST-TS commutes with S then ST-TS is nilpotent.
5. For all A,B£F_n , i) (A')'=A ii) (A+B)'=A'+B' iii) (AB)'=B'A'
6. If T£A(V) is such that (UT,V)=0 & v£V then T=0
7. If T£A(v) them v£V, there exists an element w£ v dependent on v and T such that (uT,v)=(u,w) for all u£ V
8. If T£A(v) then T*£A(v) then i) $(T^*)^*=T$ ii) $(S+T)^*=S^*+T^*$ iii) $(\lambda S)^*=\lambda S^*$ iv) $(ST)^*=T^*S^*$
9. T£A(v) is unitary iff $TT^*=1$
10. If S£A(v) and if USS*=0 then US=0
11. If T is Hermitian vT^k=0 for K≥ 1, then VT=0.
12. If N is normal Linear transformation and if UN=0 for u£V then UN*=0.
13. If N is normal and if UN^k=0 then UN=o
14. If λ & u are distinct characteristic roots of N then VN=Vλ, WN=uw & (u,w)=0
15. If N is normal & AN=NA then AN*=N*A

Sec-B

1. The Linear transformation T on V is unitary if and only if it takes an orthonormal basis of V into an orthonormal basis of V.
2. If $\{v_1, v_2, \ldots. v_n\}$ is an orthonormal basis of V and if T£A(V) in this basis (α_{ij}) then m(T*) is $\left(\beta_{ij}\right) = \overline{\alpha_{ij}}$
3. If T£A(v) is Hermitial then all its characteristic roots are real.
4. If N is normal then UNU^{-1}(UNU*) is diagonal where V is a unitary matrix.
5. The normal transformation N is
   (i) Hermitian iff its characteristic roots are real
   (ii) Unitary iff its characteristic roots are all of absolute value 1.
6. State and Prove Sylvester's law