D.K.M COLLEGE FOR WOMEN (AUTONOMOUS),VLR-1

DEPARTMENT OF MATHEMATICS

CLASS : II M.SC(MATHEMATICS)

SUBJECT : FUNCTIONAL ANALYSIS **SUB.CODE :15CPMA4C**

UNIT – I SECTION-A 6 MARKS

1. Let 'N' be a non-zero normed linear space, and Prove that 'N' is a Banach space if and only if {x:$\|x\|$ = 1} is complete.
2. State and Prove "Cauchy's Inequality".
3. Let N and N' be normed linear space and T be a linear transformation of N into N'. Then the following conditions on T are all equivalent to one another,
   (i). T is continuous.
   (ii).T is continuous at the origin, in the sense that $x_n \rightarrow 0 = T(x_n) \rightarrow 0$.
4. State and Prove "Holder's Inequality".
5. State and Prove "Minkowski's Inequality".
6. If N and N' are normed linear space, then the $\mathbb{B}(N, N')$of all continuous linear transformation of N into N'is itself a normed linear space with respect to the point wise linear operations and the norm defined by $\|T\| = Sup\{\|T(x)\|: \|x\| \leq 1\}$. Further, if N' is a Banach space, then $\mathbb{B}(N, N')$ is also a Banach space.
7. If M is a closed linear subspace of a normed linear space N, and if T is the natural mapping of N onto N/M defined by $T(x) = x + m$, show that T is a continuous linear transformation for which $\|T\| \leq 1$.
8. Let M be a linear subspace of a normed linear space N, and Let f be a functional defined on M. If $x_0$is a vector not in M, and if $m_0 = M + [x_0]$.
9. If N is a normed linear space and $x_0$is a non-zero vector in N, then there exists a functional $f_0$in N* such that $f_0(x_0) = \|x_0\| \, and \, \|f_0\| = 1$.

UNIT-II

1. A one-to-one continuous linear transformation of one Banach space onto another is a homomorphism. In particular, if a one-to-one linear transformation T of a Banach space onto itself is continuous, then its inverse T^{-1} is automatically continuous.
2. If P is a projection on a Banach space B, and if M and N are its range and null space, then M and N are closed linear subspaces of B such that $B = M \oplus N$.
3. Let B be a Banach space, and Let M and N be closed linear subspace of B such that $B = M \oplus N$. If z = x+y is the unique representation of a vector in B as a sum of vectors in M and N, then the mapping P defined by P(z) = x is a projection on B whose range and null space are M and N.
4. If B and B' are Banach spaces, and if T is a linear transformation of B into B', then T is continuous if and only if its graph is closed.
5. State and Prove “The Closed graph Theorem”.
6. Let B be a Banach space and N be a normed linear space. If $\{T_i\}$ is a non-empty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}$ is a bounded set of N for each vector x in B, then $\{\|T_i\|\}$ is a bounded set of numbers; that is $\{T_i\}$ is bounded as a subset of $\mathbb{B}(B, N)$.
7. A non-empty subset x of a normed linear space N is bounded if and only if f(x) is a bounded set of numbers for each f in N*.
8. Let B be a Banach space and N be a normed linear space. If $\{T_n\}$ is a sequence in $\mathbb{B}(B, N)$ such that $T(x) = \lim T_n(x)$ exists for each x in B. Prove that T is a continuous linear transformation.
9. State and Prove “The Uniform Boundedness Theorem”.
10. Let T be an operator on a Banach space B. Show that T has an inverse T^{-1}if and only if T* has an inverse $(T^*)^{-1}$, and that in this case $(T^*)^{-1} = (T^*)^{-1}$.
11. If x and y are any two vectors in a Hilbert space, then $|(x, y)| \leq \|x\|\|y\|$.
12. State and Prove “The Cauchy’s Schwartz’s Inequality”.
13. A closed convex subset of C of a Hilbert space H contains a unique vector of smallest norm.
14. State and Prove “The Parallelogram Law”.

15. If B is a Complex Banach Space whose norm obeys the Parallelogram Law, and if an inner product is defined on B by $4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$, then B is a Hilbert space.
16. State and Prove "The Pythagorean Theorem".
17. Let M be a Closed linear subspace of a Hilbert Space H, Let x be a vector not in M, and let d be the distance from x to M. Then there exists a unique vector $y_0$in M such that $\|x - y_0\| = d$.
18. If M is a proper closed linear subspace of a Hilbert Space H, then there exists a non-zero vector z_0 in H such that $z_0 \perp M$.
19. If S is a non-empty subset of a Hilbert Space, show that $S^{\perp} = S^{\perp\perp\perp}$.
20. If M is a linear subspace of a Hilbert Space, show that M is closed if and only if $M = M^{\perp\perp}$.
21. If $\{e_i\}$ is an Orthonormal set in a Hilbert Space H, then the set $S = \{e_i : (x, e_i) \neq 0\}$ is either empty or countable.
22. Every non-zero Hilbert Space contains a complete Orthonormal set.

UNIT-III

1. Let H be an Hilbert space, and let f be an arbitrary functional in H*. Then there exists a unique vector y in H. such that f(x)=(x,y) for every x in H.
2. Let T be an operator on Hilbert space H. Then there exist a unique operators T* on H exist a unique operators T* on H such that (Tx,y)=(x,T*y) for all x,y∈H.
3. If P is a projection on a closed linear subspace M of H if and only if I-P is a projection on

   $M^{\perp}$.

4. If 0 & I be zero and identity operator on Hilbert space H then show that $0^*=0$, I $=I^*$. Hence prove that 'T' is a non-singular then $(T^*)^{-1}=(T^{-1})^*$.
5. Show that the mapping Ψ :H→ H* defined by $\Psi(y)=f_y$ where $f_y(x)$ =(x,y) for every x∈H is 1-1,onto,additive & isometrybut not linear.
6. H** is the second conjugate operator of H is also a Hilbert space.
7. Every Hilbert space is reflexive .
8. If A_1 & A_2 are self –adjoint operator on H then the product A_1,A_2 is self-adjoint iff $A_1A_2= A_2 A_1$.

9. If T is an operator and if $(T_x,x)=0$ for all $x\in H$ for all $T=0$.
10. An operator T on H is unitary iff is an isometric isomorphism of H onto itself.
11. A closed linear subspace M of H reduces an operator T iff M is invariant under both T & T*

UNIT-IV

1. If $\|xy\| \leq \|x\|\|y\|$ shows that multiplication is jointly continuous is any banach algebra. i.e., If $x_n \to x$ and
$y_n \to y$. Then $x_n y_n \to xy$.
2. If $G = \{g_1, g_2, \ldots\ldots., g_n\}$ is a finite group then its group algebra $L_1(G)$ is the set of all complex function defined on G.
3. Show that $a \to m_a$ where $m_a(x) = ax$ is an isomorphism of a into (A).
4. G is an open set and therefore S is a closed set.
5. The mapping $x \to x^{-1}$ of G into G is continuous and is therefore a homeomorphism of G onto itself.
6. If I is a prover closed two sided ideal in A, then the Quotient algebra A.
7. Z is a subset of S.
8. The self adjoint operator in B(H) form a closed real linear subspace of B(H) and therefore real banach space which contains an identity transformation.
9. If N_1 & N_2 are normal operator on H with property that either commutes with adjointof other then N_1+N_2 & $N_1.N_2$ are normal.
10. If A is a positive operator on H .Then)I+A) is a non-singular for any arbitrary T on H.
11. If N is normal operator on H then $\|N^2\|=\|N\|^2$.
12. The boundary of S is a subset of Z
13. Every maximal left ideal in A is closed.
14. If I is a proper closed two sided ideal in A, then the quotient algebra A/I is a banach algebra.
15. If A is a division algebra , then its equals the set of all scalar multiplies of the identity

UNIT-V

1. If A/R is a semi simple banach algebra.
2. The Gelfand mapping x → $\hat{x}$ is a norm-decreasing (and therefore continuous) homorphism of A into C(m) with the following properties.
   i) The image $\bar{A}$ of A is a subalgebra of C(m) which separates the points of m and contains the identity of C(m).
   ii) The radical R of A equals the set of all elements x for which $\hat{x} = 0$, so x→ $\hat{x}$ is an isomorphism iff A is semi_simple.
   iii) An element x in A is regular iff it does not belong to any maximal ideal ⇔ $\hat{x}$(M)≠0 for every M.
   iv) If x is an element of A then its spectrum equals the range of the function $\hat{x}$ and its spectral radius equals the norm of $\hat{x}$, that is, σ(x) = $\hat{x}$(m) and r(x)=sup$|\hat{x}(m)|$=$\|\hat{x}\|$.

3. Show that M is compact Hausdroff space.

4. If f_1 and f_2 are multiplicative functional on A with the same null space M then $f_1 = f_2$.

M→f_M is a one to one mapping of the set m of all maximal ideals in A onto the set of all its functionals .

5. The maximal ideal space m is a compact Hausdroff space

6. If A is self-adjoint, then $\hat{A}$ is dense in c(m).

7. If A is self-adjoint and if $\|x^2\|$ =$\|x\|^2$ for every x, then the Gelfand mapping x → $\hat{x}$ is an isometric isomorphism of A onto c.

UNIT-I SECTION-B 15 MARKS

1) 1. Let M be a closed linear subspace of a normed linear space N if the norm of a coset $x + M$ in the quotient space N/M is defined by $\|x + M\| = inf\{\|x + M\|: m \in M\}$, then N/M is a normed linear space. Further, if N is a Banach space, then so is N/M.

2) Let a banach space B be a direct sum of a linear subspace M, N so that $B = M \oplus N$. If z = x + y is the unique expression to a vector z in B has the sum of vectors x and y in M and N then a new norm ca be defined on the linear space B by $\|z\|' = \|x\| + \|y\|$.
3) Let N and N' be normed linear transformation of N into N'. Then the following conditions on T are all equivalent to one another:
   i) T is continuous
   ii) T is continuous at the origin in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$;
   iii) There exist a real number $k \geq 0$ with the property that $\|T(x)\| \leq k\|x\| \; \forall x \in N$.
   i.e., T is bounded.
   iv) If $S = \{x/\|x\| \leq 1\}$ is a closed unit sphere in N then its image T(S) is a bounded set in N'.
4) State and prove "Hahn Banach Theorem."
5) State and prove Application of Hanbanach theorem.
6) State and prove Natural imbedding theorem.
7) Let N be a normed linear space then each vector x in N induces a functional F_x in N^* defined by $F_x(f) = f(x) \; \forall f \in N^*$ such that $\|F_x\| = \|x\|$. The mapping T: N → N^{**} such that T(x) = $F_x \; \forall x \in N$ gives an isometric isomorphism of N into N^{**}.

UNIT-II

1. State and prove Open mapping theorem.
2. If B and B′ are banach spaces and if T is a continuous linear transformation of B onto B′. Then T is open mapping.
3. Let S(x, r) be an open sphere with centre at the origin and radius r, then i) S(x, r) = x + S_r
   ii) $S_r = rS_1$.
4. State and prove application of open mapping theorem.
5. A one-to-one continuous linear transformation of one Banach space onto another is a homeomorphism. In particular, if a one-to-one linear transformation T of a Banach space onto itself is continuous, then its inverse T^{-1} is automatically continuous.

6. If B is a complex banach space whose norm obvious the parallelogram law, and if an inner product is defined on B by

   4(x, y) = $\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$. Then B is a Hilbert space.

7. If S_1, S, S_2 are non-empty subset of Hilbert space H we have i) $\{0\}^\perp = H$.

   ii) $H^\perp = \{0\}$.

   iii) S$\cap S^\perp \subset \{0\}$ iv) $S_1 \subset S_2 \Rightarrow S_2^\perp \subset S_1^\perp$ v) S$\subset S^\perp$

8. State and prove Bessel's inequality for finite set.
9. Let $\{e_1, e_2, \ldots\ldots\ldots, e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is a any vector in H. Then $\sum_{i=1}^{n} |(x, e_i|^2 \le \|x\|^2$. Further more, $x - \sum_{i=1}^{n}(x, e_i)e_i \perp e_j \ \forall j$.
10. Let $\{e_i\}$ be a finite orthonormal set in a Hilbert space H. If x is a any vector in H. Then $x - \sum_{i=1}^{n}(x, e_i)e_i \perp e_j \forall j$.

UNIT-III

1. State and prove Riesz Representation theorem..
2. The adjoint operator T→T* on B(H) as the following properties.
   a. (T1+T2)*=T1*+T2*
   b. $(\alpha$T)*=$\bar{\alpha}$T*
   c. $(T_1 + T_2)$*= T_1*T_2*
   d. T**=T
   e. $\|T\| \le \|T^*\|$
   f. $\|T^*\| < \|T\|^2$

3. Let y be a fixed vectors in aHilbert space H and let f_y be a function defined as $f_y(x)$ = (x,y) for every x∈H then f_y is functional on H & $\|f_y\| = \|y\|$.
4. If P is a projection on a Hilbert space H then (i) P is a positive on H . (ii). 0 ≤ P ≤ 1. (iii). ‖Px‖ ≤ ‖X‖ for every x∈H .(IV). ‖P‖ ≤ 1.
5. The ste of all normal operator on a Hilbert space is a closed subset of B(H) which contains the self-adjoint operator ans it is closed under scalar multiplication.

6. If P is a projection on a Hilbert space H with large M and a null space N then M⊥N iff P is self adjoint in this case N =$M^{\perp}$.
7. If $P_1,P_2 \ldots\ldots P_n$ are projectionclosed linear subspaces $M_1,M_2 \ldots\ldots M_n$ on H then P = $P_1,P_2 \ldots\ldots P_n$ iff P_i 's are pairwise orthogonal i.e p_i,p_j =0 where i ≠ j and in this case P is a projection on M where M = $M_1+M_{2+}\ldots\ldots+M_{n.}$
8. State and prove :Uniform continuity Theorem ".

UNIT-IV

1. Prove: $\sigma(x^n) = \sigma(x)^n$.
2. If A is a banach subspace of a banach algebra A' then the spectrum of an element x in A w.r.to A and A'are related as follows (i). σA'(x) ⊆ σA(x) (ii). Each boundary point of σA(x) is also boundary point of σA'(x).
3. σ(x) is non-empty.
4. $(x) = \lim \|x^n\|^{1/n}$.
5. *The* mapping x→x^{-1} of G into G is continuous and is a homeomorphism of G onto itself.

UNIT-V

1. If M is a maximal ideal in A, then the Banach algebra A/M is a division algebra, and therefore equals the Banach algebra C of complex numbers. The natural homomorphism x → x + M of A onto A/M = C assign to each element x in A complex number x(M) defined by x(M) = x + M and the mapping x → x(M) has the following properties:
   i. (i) (x + y)(M) = x(M) + y(M) (ii). (αx)(M) = αx(M) (iii). x(M) = 0 ⇔ x∈ M (iv) 1(M) = 1 (v). $|x(M)| \leq \|x\|$
2. The following conditions on A are all equivalence to one another:

   i) $\|x^2\|$ =$\|x\|$2 for every x ii) r(x)=$\|x\|$ for every x iii) $\|\hat{x}\|$ =$\|x\|$ for every x.

3. State and prove Gelfand-Neumark Theorem.