D.K.M COLLEGE FOR WOMEN (AUTONOMOUS),VELLORE-1.

DEPARTMENT OF MATHEMATICS

MAJOR: LINEAR ALGEBRA-15CMA5A

III B.SC., MATHEMATICS

UNIT I SECTION-A 2 MARKS

1. Define vector space homomorphism.
2. Define basis of a vector space.
3. Define linear span of a vector space
4. Define linear space
5. Define basis of a vector space
6. In a vector space show that $\alpha(v-w)=\alpha v-\alpha w$
7. If V is a vector space over F then $\alpha 0=0,$ for $\alpha \in \mathrm{F}$
8. Define Quotient space of V by W
9. If $v\neq 0$ then $\alpha v=0, \Rightarrow \alpha=0$
10. Define internal and external direct sum

SECTION-B 5 MARKS

1. Union of two subspaces is a subspace if and only if one is contained in the other
2. Define $L(S)$ & prove that $L(S)$ is a subspace of V.
3. If V is a vector space over F then (i) $\alpha 0=0,$ for $\alpha \in \mathrm{F}$ (ii) $0v=0,$ for $v\in \mathrm{F}$
   (iii) $(-\alpha)v=-(\alpha v),$ for $\alpha \in \mathrm{F}$ (iv) If $v\neq 0$ then $\alpha v=0, \Rightarrow \alpha=0$
4. Prove that the kernel of a homomorphism is a subspace
5. If $v_1, v_2, ..., v_n$ are in V then either they are linearly independent or some v_k is a linear combination of the proceeding ones $v_1, v_2, ..., v_{k-1}$.
6. If $v_1, v_2, ..., v_n$ are linearly independent then every element in their linear span has a unique representation in the form $\lambda_1 v_1+\lambda_2 v_2+.....+\lambda_n v_n$ with $\lambda_i \in F$
7. If $v_1, v_2, ..., v_n$ is a basis of V over F and if $w_1, w_2, ..., w_m$ in V are linearly independent over F then $m\leq n$
8. If V is finite-dimensional over F and if $u_1, u_2, ..., u_m \in V$ are linearly independent then we can find vectors $u_{m+1}, u_{m+2}, ..., u_{m+r} \in V$ such that $u_1, u_2, ..., u_m, u_{m+1}, u_{m+2}, ..., u_{m+r}$ is a basis of V
9. If A and B are finite dimensional subspaces of a vector space V then A + B is finite dimensional and dim(A+B) = dimA + dimB – dim $(A\cap B)$
10. If V is finite dimensional over F then any two bases of V have the same number of elements.

SECTION-C 10 MARKS

1. Two finite dimensional vector spaces over the same field are isomorphic if and only if they are of same dimension.
2. If V is finite dimensional & if W is a subspace of V then W is finite dimensional then $\dim W \leq \dim V$ & $\dim \frac{V}{W} = \dim V - \dim W.$

UNIT II SECTION-A 2 MARKS

1. If $\dim_F V = m$ then $\dim_F Hom(V,V) = ?$
2. Prove that $\|\alpha v\| = |\alpha| \, \|v\|$
3. Define Inner product space of a vector space V
4. Define dual space of a vector space
5. Define orthogonal complement of a subspace W of an inner product space of V
6. Define Annihilator of W
7. State Schwartz inequality
8. Define orthogonal and orthonormal set
9. Prove that $< u, \alpha v + \beta w > = \overline{\alpha} < u, v > + \overline{\beta} < u, w >$

SECTION-B 5 MARKS

1. State and prove second isomorphism theorem.
2. If V is a finite dimensional vector space over F then $V \approx \hat{\hat{V}}$.
3. Prove that $A(A(W)) = W$.
4. If a, b, c are real numbers such that $a > 0$ & $a\lambda^2 + 2b\lambda + c \geq 0, \forall$ real number λ then $b^2 \leq ac$.
5. Prove that Hom(V,W)n is a vector space over F
6. If V is finite dimensional and $v \neq 0 \in V$ then there is an element $f \in \hat{V}$ such that $f(v) \neq 0$
7. In the system of homogeneous linear equations:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0$$
$$a_{31}x_1 + a_{32}x_2 + ... + a_{3n}x_n = 0$$
$$.........$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = 0 \text{ with } a_{ij} \in F$$

Is a rank of r then there are n-r linearly independent solutions in F^n

8. Derive Schwartz inequality

9. Prove that $W^{\perp}$ is a subspace of V

10. If $\{v_i\}$ is an orthonormal set then the vectors in $\{v_i\}$ are linearly independent. If $w = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$ then $\alpha_i = (w, v_i)$ for i=1,2,….n

11. If $u, v \in V$ and $\alpha, \beta \in$ F then prove that

$$(\alpha u + \beta v,\ \alpha u + \beta v) = \alpha\bar{\alpha}(u,\ v) + \alpha\bar{\beta}(u,\ v) + \bar{\alpha}\beta(u,\ v) + \beta\bar{\beta}(u,\ v)$$

12. If $\{v_1, v_2, ...v_n\}$ is an orthonormal set in V and if $w \in V$ then $u - (w,\ v_1)v_1 - (w,\ v_2)v_2 - ..... - (w,\ v_i)v_i - ..... - (w,\ v_n)v_n$ is orthogonal to each of $\{v_1, v_2, ...v_n\}$

13. If V is a finite dimensional inner product space and if W is a subspace of V then $V = W + W^{\perp}$

14. If V is a finite dimensional inner product space and if W is a subspace of V then $(W^{\perp})^{\perp} = W$

TEN MARKS QUESTIONS

1. If $V \& W$ are of dimensions $m \& n$ respectively over F then prove that $Hom(V, W)$ is of dimension mn.
2. Prove that every inner product space has an orthonormal set as a basis.
3. Let V be the set of all continuous complex-valued functions on the closed interval [0, 1]. If $f(t),\ g(t) \in V$, define $(f(t),\ g(t)) = \int_0^1 f(t)\ \overline{g(t)}\ dt$ verify that this defines an inner product space
4. Let F be the real field and let V be the set of polynomial in a variable x over F of degree 2 or less. In V we define on inner product by if $p(x),\ q(x) \in V$ then $(p(x),\ q(x)) = \int_{-1}^{1} p(x)\ q(x)\ dx$ with the basis $v_1 = 1, v_2 = x,\ v_3 = x^2$ of V. Construct an orthonormal set.

UNIT -III SECTION-A 2 MARKS

1. Define an Algebra
2. Define Regular
3. Define characteristic vector of a vector space V
4. Define characteristic root of V
5. Define minimal polynomial for T
6. Define range of T
7. If V is a finite dimensional over F for $S,T \in A(V)$ then $r(ST) \le r(T)$

SECTION-B 5 MARKS

1. If $\lambda \in F$ is a characteristic root of $T \in A(V)$, then for any polynomial $q(x) \in F[x]$, $q(\lambda)$ is a characteristic root of $q(T)$
2. If V is a finite dimensional over F, then prove that $T \in A(V)$ is regular iff T is onto
3. If V is a finite dimensional over F for $S,T \in A(V)$ then

   (i) $r(ST) \le r(T)$ (ii) $r(TS) \le r(T)$

4. If $T \in A(V)$ and if $\dim_F^V = n$ and if T has n distinct characteristic roots in F then there is a basis of V over F which consists of characteristic vectors of T
5. If $T \in A(V)$ and if $\dim_F^V = n$ and then T can have at most n distinct characteristic roots in F
6. If $\lambda_1, \lambda_2, ...\lambda_k \in F$ are distinct characteristic roots of $T \in A(V)$ and if $v_1, v_2, ...v_k$ are characteristic vectors of T belonging to $\lambda_1, \lambda_2, ...\lambda_k \in F$ respectively then $v_1, v_2, ...v_k$ are linearly independent over F
7. If $\lambda \in F$ is a characteristic root of $T \in A(V)$ then λ is a root of the minimal polynomial of T. In particular, T has only a finite number of characteristic roots in F
8. Show that the element $\lambda \in F$ is a characteristic root of $T \in A(V) \Leftrightarrow \exists\ v \ne 0 \in V$ such that $\upsilon T = \lambda v$
9. If V is finite dimensional over F and if $T \in A(V)$ is right invertible then it is invertible
10. If V is finite dimensional over F and if $T \in A(V)$ is right invertible then T^{-1} is a polynomial expression in T over F

SECTION-C 10 MARKS

1. If A is an algebra, with unit element over F then prove that A is isomorphic to a subalgebra of
   $A(V)$ for some vector space V over F
2. If V is a finite dimensional over F, then prove that $T \in A(V)$ is invertible iff the constant term of the minimal polynomial for T is not zero
3. If V is finite dimensional over F, show that $T \in A(V)$ is regular iff T maps V onto V

UNIT IV SECTION-A 2 MARKS

1. Define matrix of $T \in A(V)$
2. Define similar transformation
3. Define invariant under $T \in A(V)$
4. Suppose that $m(s) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $m(T) = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ then find $m(ST)$

SECTION-B 5 MARKS

1. Let F be a field and Vbe the set of all polynomial in x of degree $\leq n-1$. Let D be defined by $\left(\beta_0 + \beta_1 x + \ldots\ldots.. + \beta_{n-1} x^{n-1}\right) D = \beta_1 + 2\beta_2 x + \ldots\ldots.. + (n-1)\beta_{n-1} x^{n-2}$. Compute the matrix of D in the basis $u_1 = 1,\ u_2 = 1 + x,\ \ u_3 = 1 + x^2, \ldots\ldots.. u_n = 1 + x^{n-1}$
2. If V is n-dimensional vector space over F, $A(V)$ and F_n are isomorphic as algebra's over F. Given any basis $v_1, v_2, \ldots., v_n$ of V over F if for $T \in A(V)$, $m(T)$ is the matrix of T in the basis $v_1, v_2, \ldots., v_n$ the mapping $T \to m(T)$ provides an algebra isomorphism of $A(V)$ onto F_n
3. If V is n-dimensional over F and if $T \in A(V)$ has the matrix $m_1(T)$ in the basis $v_1, v_2, \ldots., v_n$ and the matrix $m_2(T)$ in the basis $w_1, w_2, \ldots., w_n$ of V over F, then there is an element $C \in F_n$ such that $m_2(T) = C m_1(T) C^{-1}$. In fact if S is the linear transformation of V defined by $v_i s = w_i$, $i = 1, 2, \ldots.. n$ then C can be chosen to be $m_1(s)$
4. Let $V = F^3$ and suppose that $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ is the matrix of $T \in A(V)$ in the basis

   $v_1 = (1,\ 0,\ 0)$, $v_2 = (0,\ 1,\ 0)$, $v_3 = (0,\ 0,\ 1)$. Find the matrix T in the basis

5. If V is n-dimensional over F and if $T \in A(V)$ has a characteristic roots in F then T satisfies a polynomial of degree n over F

SECTION-C 10 MARKS

1. (i) If $W \subset V$ is invariant under T then T induces a linear transformation $\overline{T}$ on V/W defined

   by $(v+W)\overline{T} = \bar{v}\,\overline{T} = vT + W$

   (ii) If T satisfies the polynomial $q(x) \in F[x]$ then so does $\overline{T}$ also satisfies $q(x)$

   (iii) If $p_1(x)$ is the minimal polynomial for $\overline{T}$ over F and if $p(x)$ is that for T then $p_1(x) \;/\; p(x)$
2. If V is n-dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, then prove that there is a basis of V in which the matrix of T is triangular

UNIT V SECTION-A 2 MARKS

1. Prove that tr $(A + B)$ = tr$A$ + tr$B$
2. If F is the field of complex numbers, than evaluate the determinant $\begin{vmatrix} 1 & i \\ 2-i & 3 \end{vmatrix}$
3. Define trace and transpose of the matrix A
4. Define symmetric and skew-symmetric matrix
5. Define determinant of A
6. Define even and odd permutation
7. Prove that det A^{-1} = (det A)$^{-1}$
8. Define secular equation
9. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ then find its secular equation
10. State cramer's rule
11. If A is invertible then for all B, det(ABA^{-1}) = det B
12. If A is invertible thentr(ACA^{-1}) = trC

SECTION-B 5 MARKS

1. If F is of characteristic 0 and if S and T, in $A_F(V)$ are such that $ST - TS$ commute with S, then $ST - TS$ is nilpotent
2. For A, B $\in F_n$ and $\lambda \in F$ (i) tr (λA) = λtr(A) (ii) tr(A + B) = trA + trB (iii) tr(AB) = tr(BA)

3. If F is a field of characteristic zero and if $T \in A(V)$ is such that tr$T^i = 0 \; \forall i \geq 1$ prove that T is nilpotent
4. Prove that the determinant of a triangular matrix is the product of its entries on the main diagonal
5. If two rows of A are equal (ie., $v_r = v_s$, for $r \neq s$) then detA = 0
6. Interchanging two rows of A changes the sign of its determinant
7. A is invertible if and only if det A $\neq 0$
8. Every A $\in F_n$ satisfies its secular equation
9. Det A is the product, counting multiplicities of the characteristic roots of A

SECTION-C 10 MARKS

1. For $A, B \in F_n$, prove that det (AB) = (det$A$) (det$B$)
2. State and prove cramer's rule
3. Prove that detA = detA1 where A1 is the transpose of A
4. For all A, B $\in F_n$ then prove that (i) $(A^1)^1 = A$ (ii) $(A + B)^1 = A^1 + B^1$ (iii) $(AB)^1 = B^1A^1$