D.K.M. COLLEGE FOR WOMEN (AUTONOMOUS),VELLORE-1.

DEPARTMENT OF MATHEMATICS

I MSC MATHEMATICS

PARTIAL DIFFERENTIAL EQUATIONS

SUBJECT CODE:15CPMA2C

UNIT-I **SECTION-A**

1. (a) Solve $f(x+y+z, x^2+y^2+z^2)=0$.

   (b) Form partial differential equation by eliminating arbitrary constant $Z=ax+by+ab$.

2. (a) Solve $y^2zp+x^2zq=y^2x$.

   (b) Solve $xzp+yzq=xy$.

3. Find the integral surface of the linear PDE containing the straight line $x+y=0$, $z=1$ $x(y^2+z)P-y(x^2+z)q=(x^2-y^2)z$.

4. Explain the Cauchy problem for first order equation.

5. (a) Find the characteristic equation of PDE $p^2+q^2=2$ and determine the integrables surface which passes through $x=0$, $z=y$.

   (b) Along every strip(characteristic strip) of the PDE $F(x,y,z,p,q)=0$ then the function $F(x,y,z,p,q)$ is constant.

6. Derive the charpitz method.

7. Using charpitz equation find the complete integral of $Z^2=pqxy$.

8. Find the characteristic equation of the $pq=z$ and determine the integral surface which passes through the straight line $x=1;z=y$.

UNIT-II

9. Derivation of Laplace Equation and poisson Equation.

10. Derive the Exterior Dirichlet problem for a circle.

11. Derive the Interior Newmann problem for a circle.

12. Derive the solution of Laplace Equation in Cyclindrical Co-ordinates.

13.Solve $\nabla^2 u=0$, $0\le x\le a$, $0\le y\le b$, satisfying boundary conditions u(0,y)=0, u(x,0)=0, u(x,b)=0, $\frac{\partial u}{\partial x}(a,y)=T\sin\frac{3\pi y}{a}$.

SECTION –B UNIT-I

1. The general solution of a linear PDE Pp+Qq=R can be written in the form F(u,v)=0, where F is an arbitrary function and $v(x,y,z)=c_2$, u(x,y,z)=0 form a solution of the equation.

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{Z(x,y,z)}.$$

2. Derive the necessary and sufficient for PDE are compatible.

3. Solve the following PDE xp-yq=x and $x^2p+q=xz$ are compatible and hence find the equation.

4. Derive the canonical Form.

5. State and Prove the Riemann method.

6. Obtain the Riemann's Solution for the equation $\frac{\partial^2 u}{\partial x\partial y}=F(x,y)$. Given

(i) u=F(x) on Γ

(ii) $\frac{\partial u}{\partial \eta}$=g(x) on Γ

Where Γ is the curve y=x,

UNIT-II

1. Derive the Dirichlet's problem for a Rectangle.
2. Derive the Newmann problem for a Rectangle.
3. Derive the Interior Dirichlet problem for a circle.
4. Derive the solution of Laplace Equation in spherical Co-ordinates.
5. In a soiid sphere of radius 'a' the surface is maintained at the temperature iven by,
   i. $f(\theta)=\begin{cases} k\cos\theta, 0\le\theta\le\frac{\pi}{2} \\ 0, \frac{\pi}{2}\le\theta\le\pi \end{cases}$
6. 12. Prove that the steady state temperature with in the solid is,
7. $U(r,\theta)=k[\frac{1}{2}P_0(\cos\theta)+\frac{1}{2}(\frac{r}{a})P_1(\cos\theta)+\frac{5}{16}\left(\frac{r}{a}\right)^2P_2(\cos\theta)-\frac{3}{32}\left(\frac{r}{a}\right)^4P_4(\cos\theta)+\cdots]$

UNIT III SECTION-A

1. Definition of Boundary Conditions.
2. Elementary solution of the Diffusion Equation (or) Derive the solution of diffusion Equation (or) Consider the one dimensional diffusion equation $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \longrightarrow \quad (1) \quad -\infty < x < \infty\,, \; t > 0$. The function $T(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} e^{\frac{-(x-\xi)^2}{4\alpha t}}$ Where ξ is an arbitrary real constant is a solution of equation (1).
3. In one dimensional infinite solid $-\infty < x < \infty$ is initially maintained at temperature and at zero temperature everywhere outside the surface. Such that $T(x,t) = \frac{T_0}{2}\left[erf\left(\frac{b-x}{\sqrt{4\alpha t}}\right) - erf\left(\frac{a-x}{\sqrt{4\alpha t}}\right)\right]$ where erf is an error function.
4. Dirac Delta Equation and Dirac Delta Functions.
5. Important properties of Dirac Delta Functions.
6. Separations of Variables Method.
7. Determine the temperature $T(r,t)$ in the infinite cylinder $0 \le r \le a$ when the initial temperature is $T(r,0)=f(r)$ and the surface $r=a$, is maintained at a temperature.
8. Find the temperature in a sphere of radius a, when its surface is kept at zero temperature and its surface is kept at zero temperature and its initial temperature is $f(r,\theta)$.
9. Find the solution of the one dimensional Diffusion Equation satisfying the following Boundary Conditions,
   - *(i)* T is bounded as $t \to \infty$.
   - *(ii)* $\frac{\partial T}{\partial x}$ at $x=0, \; \forall\, t$.
   - *(iii)* $T(x,0) = x\,(a - x), \quad 0 < x < a$.

SECTION-B

1. Derivation of Heat Equation from a basic concept (or) Derive the Fourier Heat Conduction Equation.
2. Nature of Solution of the Heat Equation.
3. Solve the one dimensional diffusion Equation in the region $0 \le x \le \pi,\; t \ge 0$ subject to the condition.
   - *(i)* T remains finite as $t \to \infty$.
   - *(ii)* $T=0$ if $x=0$ and $x=\pi$ for all t.

(iii) At $t=0$, $T=\begin{cases} x, & 0 \le x \le \frac{\pi}{2}. \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$

4. A Uniform rod of length L whose surface is thermally insulated is initially at temperature $\theta = \theta_{0,}$ at time $t=0$ one end is suddenly cooled at $\theta = 0$ and subsequently maintained at this temperature. The order end remains thermally insulated. Find the temperature distribution $\theta(x, t)$.
5. Solution of diffusion equation in Cylindrical Co-ordinates.
6. The conducting bar of uniform cross section lies along the x axis with ends at $x=0$ and $x=L$. It is kept initially at temperature $\theta°$ its lateral surface is insulated. There are no heat sources and the bar the end $x=0$ is kept at $0°$ and heat is suddenly applied at the end $x=L$. So that there is a constant flour q_0 at $x=L$. Find the temperature distribution in the bar for $t>0$.
7. The ends A and B of a rod $10cm$ is length are kept $0℃$ and $100℃$ until the steady state condition suddenly the temperature at the end A is increased to $20℃$ and the end B is decreased to $60℃$. Find the temperature distribution in the rod at a time t.
8. Assuming the surface of the earth to be flat, which is initial at zero temperature and for times $t>0$, the boundary surface is being subjected to a periodic heat flux $g_0 \cos \omega t$. Investigate the penetration of these temperature variations into the Earth's surface and such that at a depth x, the temperature fluctuates and the amplitudes of the steady temperature is given by $\frac{g_0}{\sqrt{2}}\sqrt{\frac{2\alpha}{\omega}}\, exp\left[-\sqrt{\frac{\omega}{2\alpha}}x\right]$.
9. The boundary of the rectangle $0 \le x \le a,\ 0 \le x \le b$, are maintained at zero temperature if at $t=0$. The temperature T has the prescribed value $f(x,y)$. Show that for $t>0$ the temperature at a point within the rectangle is given by, $T(x,y,z)$ $= \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)e^{-(\alpha\lambda^2 mnt)} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$ where, $F(m,n) = \int_0^a \int_0^b f(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy$ and $\lambda^2 mn = \pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{a^2}\right)$.
10. Solution of Diffusion Equation in spherical coordinates.

UNIT IV SECTION A

1. Consider Maxwell's Equation of Electromagnetic theory is given by $\nabla \cdot E = 4\pi\rho, \nabla \cdot H = 0, \nabla \times E = \frac{-1}{c}\frac{\partial H}{\partial t}, \nabla \times H = \frac{4\pi i}{c} + \frac{1}{c}\frac{\partial E}{\partial t} = 0$. Where E is an Electric field, ρ is the Electric Charge Density, H is the Magnetic Field, i is the Current

Density and c is the Velocity of light. Show that in the H satisfy the Wave Equation when $\rho = i = 0$.

2. The Initial value problem D'Alemberts.
3. A stretched string of a finite length L is hold fixed at its end and is subjected to an initial displacement $u(x,0)=u_0 \sin\frac{\pi x}{L}$. The string is released from this position with zero initial velocity. Find the result time dependent motion of the string.
4. Obtain the solution of the wave equation $u_{tt} = c^2 u_{xx}$ under the following condition.
   - *(i)* $u(0,t) = u(2,t) = 0$.
   - *(ii)* $u(x,0)= \sin^3\frac{\pi x}{2}$.
   - *(iii)* $u_t(x,0) = 0$.
5. Prove that the Total Energy of a string which is fixed as the points $x=0$, $x=L$ and executing small transverse vibrations is given by $\frac{1}{2}T\int_0^L\left[\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{C^2}\left(\frac{\partial y}{\partial t}\right)^2\right]dx$ where, $c^2 = \frac{T}{\rho}$, ρ is the uniform linear density and T is the tension, show also that if $y = f(x-ct)$, $0 \le x \le \pi$, then, the energy of the wave is equally divided between Potential Energy and Kinetic Energy.
6. Periodic solution of one dimensional wave equation in cylindrical coordinates.
7. State and prove Uniqueness Theroem.
8. State and prove Duhamel's Principle.
9. Use Duhamel's Principle to solve the heat equation $v_t(x,t) = kv_{xx}(x,t), -\infty < x < \infty, t > 0,$ $v(x,0) = 0, -\infty < x < \infty$.

SECTION B

1. Derivation of one dimensional wave equation.
2. Solution of one dimensional wave equation by Canonical Reduction.
3. Obtain the periodic solution of the wave equation in the form $u(x,t) = Ae^{i(kx+wt)}$ where $i = \sqrt{-1}$, $k \pm {}^{W}/_{c}$, A is constant and hence define various terms involved in wave propagation.
4. Solve the following initial value problem of the wave equation (Cauchy Problem) described by the Homogeneous wave equation

   PDE : $u_{tt} - c^2 u_{xx} f(x,t)$.
   IC's : $u(x,0) = \eta(x)$, $u_t(x,0) = v(x)$.
5. Vibrating string – Variable separable.

6. A string of length L is release from rest in the position y=f(x). Show that the total energy of the string is $\frac{\pi^2 T}{4L}\sum_{n=1}^{\infty} n^2 k_n^2$ where $k_n = \frac{2}{L}\int_0^L f(x)\sin n\frac{\pi x}{L}dx$, T= Tension of the string. If the midpoint of the string is pulled a side through a small distance and then released. Show that in the subsequent motion the fundamental contributes $\frac{\pi^2}{8}$ of the $+\sum(some\ multiple)$.
7. Boundary and initial value problem for two dimensional wave equation of Eigen function.
8. Periodic solution of one dimensional equation in Spherical Polar Coordinates.
9. Vibration of a Circular Membranes.

UNIT V SECTION A

1. Show that the delta function is the derivative of a heavy side unit step function *H(x)* respectively.
2. The notation of a delta function and its derivative enables us to give a meaning to the derivative if a function that has jump continuity at $x = \xi$ of magnitude unity.
3. To illustrate a differential operator by considering the boundary value problem $\frac{d^2u}{dx^2} = f(x); u(0) = u(1) = 0$
4. Concept of PDE of Higher Dimensional.
5. The concept on Green's function is defined as follows.
6. Using Green's function technique to solve the Dirichlet Problem for a Semi-infinite space.
7. Consider the case when R consists at G Half plane defined by $x \geq 0$, $-\infty < y < \infty$ and hence solve $\nabla^2 u = 0$. In the above region subject to the conditions $u = f(y)$ on $x = 0$ using the Green's function technique.
8. Eigen function method.
9. Find the Green's function for the Dirichlet problem on the rectangle $\mathbb{R}$, $0 < x < a$, $0 < y < b$ described by PDE $(\nabla^2 + \lambda)u = 0$ in $\mathbb{R} \rightarrow (1)$ and the boundary condition u = 0 on $\partial\mathbb{R}$.
10. Determine the Green's function for the Helmholtz equation for the Half space $z \geq 0$.
11. Solve the following IBVP using the Laplace transform technique

PDE : $u_t = u_{xx}$, $0 < x < 1$, $t > 0$,

BC's: $u(t,0) = 1$, $u(1,t) = 1$, $t > 0$,

IC's: $u(x,0) = 1 + \sin\pi x$, $0 < x < 1$.

12. Solve the IBVP described by

PDE : $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$,
BC's: $u(t,0) = u(1,t) = 0$, $t > 0$,
IC's: $u(x,0) = \sin \pi x$, $u(x,0) = -\sin \pi x$, $0 < x < 1$.

13. A string is stretched and fixed between two points $(0,0)$ and $(L,0)$ motion is initiated by displacing the string in form $u = \lambda \sin\left(\frac{\pi x}{l}\right)$ and realize from the rest at time $t = 0$. Find the displacement of any point on the string at any time t.

14. An infinitely long string having one end at $x = 0$ is initially at rest on the x axis the end $x = 0$ under goes a periodic transverse displacement described by $A_0 \sin \omega t$, $t > 0$. Find the displacement of any point on the string at any time t.

SECTION B

1. Derive the Green's function for Laplace Equation.
2. Obtain the solution of Interior Dirichlet problem for a sphere using the Green's function Method and hence derive the Poisson Integral Formula.
3. Determine the Green's function for the Dirichlet Problem for a circle given by $\nabla^2 u = 0$, $r < a$, $u = f(\theta)$ on $r = a$.
4. Derive the Green's function for the wave equation Helmholtz theorem.
5. Derive the Green's function for the Diffusion Equation.
6. If the function $u(x,t)$ satisfying the following:

PDE : $u_{xx} = \frac{1}{c^2} u_{tt} + k$, $0 < x < l$, $t > 0$,

BC's: $u(0,t) = 1$, $u(l,t) = 1$, $t > 0$,

IC's: $u(x,0) = u_t(x,0)$, $0 \le x \le l$.

7. Find the solution of the BVP given by

PDE : $u_{xx} = \frac{1}{k} u_t$, $0 \le x \le a$, $t > 0$,

BC's: $u(0,t) = f(t)$, $u(a,t) = 0$,

IC's: $u(x,0) = 0$, *using Laplace transformation method.*

8. Using the Laplace transform method IBVP described as

PDE : $u_{xx} = \frac{1}{c^2} u_{tt} - cos\omega t, \ \ 0 \leq x \leq \infty, \quad 0 \leq t \leq \infty,$

BC's: $u(0,t)=0$, u is bounded as $t \to \infty$.

IC's: $u_t(x,0) = u(x,0) = 0$.

9. Using the Laplace transform solve the following initial boundary value problem.

PDE : $ku_t = u_{xx}, \ \ 0 < x < 1, \ \ 0 < t < \infty$,

BC's: $u(0,t)=0, \ \ u(l,t)= g(t), \quad 0 < t < \infty,$

IC's: $u(x,0) = 0, \quad 0 \leq x \leq l.$