D.K.M COLLEGE FOR WOMEN (AUTONOMOUS),VELLORE-1

DEPARTMENT OF MATHEMATICS

CLASS : I M.SC (MATHEMATICS) **SUBJECT : REAL ANALYSIS-I**

UNIT-I SECTION-A 6 MARKS

1. If 'f' is monotonic on [a,b],then the set of discontinuous of f is countable.
2. If 'f' is monotonic on [a,b],then f is of bounded variation on [a,b].
3. If 'f' is continuous on [a,b] and,if f exists and is bounded in the interior , say A for all x in [a,b], then f is of bounded variation on [a,b].
4. Let f be bounded variation on [a,b],and assume that c (a,b),Then f is of bounded variation on [a,c] and on [c,b] and we have
5. Let f be bounded variation on [a,b],Let U be defined on [a,b] as follows: if then i)V is an increasing function on [a,b]. ii)V-f is an increasing function on [a,b].
6. Let f be defined on [a,b]. then f is bounded variation on [a,b] iff,f can be expressed as the the difference of two increasing functions.
7. To construct a continuous function which is not of bounded variation . Let f(x)= x cos $\{\pi/2x\}$ If x≠0, f(0)=0.Then f is continuous on [0,1].
8. If f is of bounded variation on [a,b], say Σ for all partition of [a,b],then f is bounded on [a,b].

UNIT-II

9. If and if on [a,b] then on [a,b] (for any two constants and) and we have $\int (C_1f+C_2g)d\alpha = C_1\int_a^b f d\alpha + C_2\int_a^b g d\alpha.$
10. If f∈ R(α) and if f ∈ R(β) on [a,b] then f∈ R (C1α+C2 β) On [a,b] (for any two constants and) and we have $\int_a^b f d(c1\alpha+c2\beta) = C_1\int_a^b f d\alpha + C_2\int_a^b g d\alpha.$
11. If f∈ R(α) on [a,b] , then $\alpha \in R(f)$ on [a,b] and we have $\int_a^b f(x)d\alpha(x) + \int_a^b \alpha(x)df(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a)$.
12. Every finite sum can be written as a Riemmann-stielties intergral,given a sum Σ , define f on [0,n] as follows: f(x) = a_k if k-1 < x≤k(k=1,2….),f(0)=0 .Then$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \int_0^n f(x)d[x]$, where [x] is the greatest integer .
13. .Euler's summation formula: If f has a continuous derivative 'f' n [a,b], then we have $\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \sum_a^b f'(x)((x))dx + f(a)((a))-f(b)((b))$, where((x))=x-[x],when a and b are integers this becomes $\int_{n=1}^{b} f(n) = \int_a^b f(x)dx + \sum_a^b f'(x)(x-[x]-1/2)dx + f(a)+f(b)/2$.

14.Assume that on [a,b] then : i) If p' is finer than p, we have U(p',f,α) ≤ U (p,f,α) and L(p',f,α) ≥ L (p,f,α)

ii)For any two partitions p_1 and p_2, we have $L(p_1,f,\alpha)$) ≤ U (p_2,f,α).
15. Assume that α on [a,b]. Then τ(f, α) ≤ τ(f, α)
16. Assume that α on [a,b]. If f R(α) and g R(α) on [a,b] if f(x) ≤ g(x) for all x in [a,b],then we have

$\int_a^b f(x)d\alpha(x) \leq \int_a^b f(x)d\alpha$.

17. Assume that α on [a,b]. If f R(α) on [a,b]. Then $f^2 \in R(\alpha)$ on [a,b].
18. Assume that α on [a,b]. If f R(α) and g R(α) on [a,b].then the product f.g R(α) on [a,b].

UNIT-III

19.Let α be of bounded variation on [a,b] and assume that f R(α) on [a,b].Then f on every sub-interval [c,d] of [a,b].
20. If f is continuous on [a,b] and if α is of bounded variation on [a,b] and assume that f R(α) on [a,b].
21. First mean – value Theorem for Riemann-Stieltjes Integrals:
22. Second mean – value Theorem for Riemann-Stieltjes Integrals:
23. Assume that f R on [a,b] . let α be a function which is continuous on [a,b]. and whose derivative α' is Riemann integrable on [a,b]. Then the following integrals exist and are equal $\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\,\alpha'(x)dx$.
24. If f R and g R on [a,b],Let F(x)= $\int_a^b f(t)dt$, G(x) = $\int_a^b g(t)dt$,if x∈[a,b]. Then F and G are continuous function of bounded variation on [a,b] .Also , f R(G) and g R(F) on [a,b] , and we have $\int_a^b f(x)g(x)dx = \int_a^b f(x)dG(x) = \int_a^b g(x)dF(x)$.
25. State and prove : Second fundamental theorem of integral calculus.
26. Assume that f R on [a,b] . let α be a function which is continuous on [a,b]. and whose derivative α' is Riemann integrable on [a,b]. Then the following integrals exist and are equal

$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\,\alpha'(x)dx$.

27. Let g be continuous Assume that f on [a,b]. Let A and B be two real number satisfying the inequalities A≤ f(a+) and B≥f(b-). Then there exists a point x_0 in [a,b] such that

(i) $\int_a^b f(x)g(x)dx = A\int_a^x g(x)dx + B\int_x^b g(x)dx$. In particular ,if f(x) ≥0 for all in [a,b] .we have

(ii) $\int_a^b f(x)g(x)dx = \int_a^b)g(x)dx$.

28. State and prove : Bonnet's T heorem.
29.If f is continuous on the rectangle [a,b] x [c,d], and if g R on [a,b],Then the function f defined by the equation F(Y) = $\int_a^b g(x)f(x,y)dx = \int_a^b g(x)f(x,y_0)dx$.
30. If f is continuous on the rectangle [a,b] x [c,d], and if g R on [a,b] and if h R on [c,d], Then we have $\int_a^b[\int_c^d g(x)h(y)f(x,y)dy]dx = \int_c^d[\int_a^b g(x)h(y)f(x,y)dx]dy$.
31. Let Q ={(x,y);a≤ x ≤ b, c≤ y ≤ d} .Assume that α of bounded variation on [a,b] and, for each fixed y in [c,d], assume that the integral $\int_a^b f(x,y)\,g(x)dx$ exists. If the partial derivative f is continuous on Q , the derivative F'(y) exist for each y in (c,d) and is given by $F'(x) = \int_a^b D_2 f(x,y)d\alpha(x)$.

32. Absolute convergence of Σ implies convergence.
33. Let Σ be a given series with rea-valued term and define
p_n = $|a_n|$ +a_n /2 , Q_n = $|a_n|$ - a_n /2 then, (i) .
If Σ is conditionally convergent,both Σ and Σ diverge.
ii) If Σ converges, both ΣP_n and Σ Qnconverges and we have $\sum_{n=1}^{\infty} p_n$ - $\sum_{n=1}^{\infty} q_n$.
34. If $\{a_n\}$ and $\{b_n\}$ are two sequences of complex number, defineA_n= a_1 + a_2 +……+a_n then we have the identity $\sum_{k=1}^{n} a_k b_k$ = $A_n b_{n+1}$-$\sum_{k=1}^{n} A_k(b_{k-1}-b_k)$. the series $\sum_{k=1}^{n} a$kbk converges if both the series $\sum_{k=1}^{n} A_K$ $(b_{k-1}-b_k)$and the sequence $\{A_n b_{n+1}\}$.
35. The series $\Sigma a_n b_n$ converges if Σa_n converges and if $\{b_n\}$ is a monotonic convergent series
36. State and prove : Dirclet's test
37. Let Σa_n be a series of complex term whose partial sums from a bounded sequence . Let $\{b_n\}$ be a decreasing sequence which converges to 0. Then $\Sigma a_n b_n$ converges.

UNIT-IV

38. Assume that For each fixe p, assume that the limit exist. Then the limit also exist and the value a.
39. Let Σa_m and Σb_n be two a ly convergent series with sums A and B respectively . Let f be the double sequence define by the equation if (m,n) = $a_m b_n$, if (m,n) $\in z^+ \times z^+$,then $\Sigma_{m,n}$ f(m,n) converges absolutely and has the sum AB.
40. The infinite product πu_n converges iff ∈>0 there exist an N such that n>N implies $|u_{n+1}, u_{n+2}, \ldots u_{n+k-1}|$ < ∈ ,for k=1,2,3…
41. Assume that each $a_n > 0$. Then the product $\pi(1+a_n)$ converges iff the series Σan converges
42. Absolutely converges of $\pi(1+a_n)$ implies converges.
43. Given a power series $\sum_{n=0}^{\infty} a$n $(z-z0)^n$, Let λ = $\lim_{n\to\infty} sup\sqrt{|a_n|}$,r=$1/\lambda$ Then the series Absolutely converges , if and diverges if$|z-z_0|$ >r, Furthermore , the series converges uniformly on every compact subset interior to the disk of convergence.
45. Assume that we havef(x) = $\sum_{n=0}^{\infty} a_n x^n$ if –r < x < r . If the series and we have $\lim_{x\to r-} f(x) = \sum_{n=0}^{\infty} a\, n r^n$.
49. Let $\sum_{n=0}^{\infty} a$n and $\sum_{n=0}^{\infty} b$n be two convergent series and let $\sum_{n=0}^{\infty} c$n denote their Cauchy product. If $\sum_{n=0}^{\infty} c$n converges, we have $\sum_{n=0}^{\infty} c$n = ($\sum_{n=0}^{\infty} a$n) $\sum_{n=0}^{\infty} b_n$)

UNIT-V

51.Assume that $f_n \to f$ uniformly on s. If each fn is continuous at a point c of s, then the limit function f is also continuous at c.

52.State and prove Cauchy Condition for Uniform Convergence of Series.

53.State and prove Weierstrass M-test.

54.Assume that $\sum f_n(x)=f(x)$ (uniformly on s). If each fn is continuous at a point x_0 of s, then f is also continuous at x_0.

55.The infinite series $\sum f_n(x)$ converges uniformly on s, iff for every $\varepsilon > 0$ there is an N such that n>N implies $|\sum_{k=n+1}^{n+p} f_k(x)| < \varepsilon$ for each p=1,2,..and every x in s.

56.Let $\{M_n\}$ be a sequence of non negative numbers such that $0 \le |f_n(x)| \le M_n$ for n=1,2,…and for every x in s. Then $\sum f_n(x)$ converges uniformly on s, if $\sum M_n$

Converges.

57.Let f be a double sequence and Let z^+ denote the set of positive integers. For each n=1,2,….,define a function gn on z^+ as follows. $G_n(m)=f(m.n)$, if $m \epsilon z^+$.Assume that $g_n \to g$ uniformly on z, where $g(m)=\lim_{n\to\infty} f(m,n)$. If the iterated limit $\lim_{m\to\infty}(\lim_{n\to\infty} f(m,n))$ exists, then the double limit $\lim_{n,m\to\infty} f(m,n)$ also exists and has the same value.

58.Assume that $\lim_{n\to\infty} f_n=f$ on [a,b]. If $g \epsilon R$ on [a,b] define $h(x)=\int_a^x f(t)(t)dt$, $h_n(x)=\int_a^x f_n(t)(t)$, if $x\epsilon[a,b]$. Then $h_n \to h$ uniformly o [a,b].

59.Assume that$\lim_{n\to\infty} f_n=f$ and $\lim_{n\to\infty} g_n=g$ on [a,b]. Define $h(x)=\int_a^x f(t)g(t)dt$, *$h_n(x)=\int_a^x f_n(t)g_n(t)dt$ if $x\epsilon[a,b]$. Then $h_n \to h$ uniformly on [a,b].*

UNIT-I SECTION-B 15 MARKS

1. *Assume that f and g are each of bounded variation on [a,b]. Then so are theiir sum,difference and product. Also we have $v f\pm g \le v f + v g$ and $v_{f.g} \le A v_f + B v g$, where $A=Sup\{|g(x)|:x\epsilon[a,b]\}$, $B=Sup\{|f(x)|:x\epsilon[a,b]\}$.*
2. *Let f be a bounded variation on [a,b]. If $x\epsilon[a,b]$.Let $v(x)=vf(a,x)$ and put $v(a)=0$. Then every point of continuity of f is also a point of continuity of U. The converse is also true.*

UNIT-II

3. *Assume that $c\epsilon(a,b)$. If two of the three intergals in exists, then the third also exists and we have $\int_a^c f dx + \int_c^b f dx = \int_a^b f dx$.*
4. *State and prove Change of variable in a Riemann Stieltjes integral.*
5. *State and prove Reduction to a Riemann integral.*
6. *Let $f\epsilon R(\propto)$ on [a,b] and let g be a strictly monotonic continuous function defined on an interval s having end points c and d. Assume that a=g(c), b=g(d). Let h and β be the composite functions defined as follows $h(x)=f[g(x)]$, $\beta(x)=\propto[g(x)]$, if $x\epsilon s$, Then $h\epsilon R(\beta)$ on s and we have $\int_a^b f d\propto = \int_c^d h d\beta$, That is,$\int_{g(c)}^{g(d)} f(t) d\propto(t) = \int_c^d f[g(x)] d\{\propto [g(x)]\}$.*
7. *Assume $f\epsilon R(\propto)$ on [a,b] and assume that $\propto$ has a continuous derivative $\propto'$ on [a,b].Then the Riemann integral $\int_a^b f(x) \propto'(x) dx$ exists and we have $\int_a^b f(x) d\propto(x) = \int_a^b f(x)\propto'(x)dx$.*
8. *Assume that $\propto\uparrow$ on [a,b]. Then the following three statements are equivalent*
   (i) *$f\epsilon R(\propto)$ on [a,b].*

(ii) *f satisfies Riemann condition with respect to* $\propto$ *on [a,b].*

(iii) $\underline{I}(f,\propto)=\bar{I}(f,\propto)$.

9. *Assume that* $\propto\uparrow$ *on [a,b]. If f*ϵ*R(*$\propto$*) and g*ϵ*R(*$\propto$*) on [a,b] and if f(x)≤g(x) for all x in [a,b], then we have* $\int_a^b f(x) \propto (x) \leq \int_a^b g(x)\, d \propto (x)$.

UNIT-III

10. *Assume that* $\propto$ *is of bounded variation on [a,b]. Let v(x) denote the total variation of* $\propto$ *on [a,x] if a<x≤b, and let v(a)=0. Let f be defined and bounded on [a,b]. If f*ϵ*R(*$\propto$*) on [a,b] then f*ϵ*R(v) on [a,b].*
11. *Assume f*ϵ*R(*$\propto$*) and g*ϵ*R(*$\propto$*) on [a,b], where* $\propto\uparrow$ *on [a,b]. Define* $F(x)=\int_a^x f(t) d \propto (t)$ *and* $G(X)=\int_a^x g(t) d \propto (t)$*, if x*ϵ*[a,b] Then f*ϵ*R(G), g*ϵ*R(F) and the product f.g*ϵ*R(*$\propto$*) on [a,b] and we have* $\int_a^b f(x)g(x) d \propto (x) = \int_a^b f(x) dG(x) = \int_a^b g(x) dF(x)$.
12. *Let* $\propto$ *be of bounded variation on [a,b] and assume that f*ϵ*R(*$\propto$*) on [a,b]. Define f by the equation* $F(x)=\int_a^x f\, d\propto$*, if x*ϵ*[a,b]. Then we have*
    (i) *F is of bounded variation on [a,b].*
    (ii) *Every point of continuity of* $\propto$ *is also a point of continuity of F.*
    (iii) *If* $\propto\uparrow$ *on [a,b], the derivative F'(x) exists at each point x in (a,b) where* $\propto'(x)$ *exists and where f is continuous. For such x, we have* $F'(x)=f(x)\propto'(x)$.
13. *Assume that g has a continuous derivative g' on an interval [c,d]. Let f be continuous on g([c,d]) and define F by the equation* $F(x)=\int_{g(c)}^x f(t)dt$ *if x*ϵ*g([c,d]). Then, for each x in [c,d] the intergal* $\int_c^x f[g(t)]g'(t)dt$ *exists and has the value F[g(x)]. In particular, we have* $\int_{g(c)}^{g(d)} f(x)dx = \int_c^d f[g(t)]g'(t)dt$.
14. *State and prove Riemann Stieltjes integrals depend in y on a parameter.*
15. *Let f be continuous at each point (x.y) of a rectangle Q={(x,y);a≤x≤b, c≤y≤d}. Assume that* $\propto$ *is of bounded variation on [a,b] and let F be the function defined on [c,d] by the equation* $F(Y)=\int_a^b f(x,y) d\propto(x)$*. Then F is continuous on [c,d]. In otherwords, if* y_0ϵ*[c,d], we have*
    $lim_{y\to y_0} \int_a^b f(x,y) d\propto(x) = \int_a^b lim_{y\to y_0} f(x,y) d\propto(x) = \int_a^b f(x,y_0) d\propto(x)$.
16. *State and prove Interchanging the order of Integration.*
17. *Let Q={(x,y); a≤ x ≤ b, c≤y≤d}. Assume that* $\propto$ *is of bounded variation on [a,b], β is of bounded variation on [c,d], and f is continuous on Q. If (x,y)*ϵ*Q define* $F(y)=\int_a^b f(x,y) d\propto(x)$, $G(x)=\int_c^d f(x,y) d\beta(y)$*. Then F*ϵ*R(β) on [c,d], G*ϵ*R(*$\propto$*) on [a,b], and we have* $\int_c^d F(y) d\beta(y) = \int_a^b G(x) d\propto(x)$*. In otherwords, we may interchange the order of interation as follows;*
    $\int_a^b [\int_c^d f(x,y) d\beta(y)] d\propto(x) = \int_c^d [\int_a^b f(x,y) d\propto(x)] d\beta(y)$.
18. *Let f be defined and bounded on [a,b] and let D denote the set of discontinuities of f in [a,b]. Then f*ϵ*R on [a,b] iff D has measure zero.*
19. *State and prove Lebesgue's Criterion for Riemann Integrability.*

UNIT-IV

20. *Let* $\sum f(m,n)$ *be a given double series and let g be an arrangement of the double sequence f into a sequence G. Then*

(a) $\sum G(n)$ converges absolutely iff $\sum f(m,n)$ converges absolutely. Assume that $\sum f(m,n)$ does coverge absolutely, with sum s, we have further.
(b) $\sum_{n=1}^{\infty} G(n)=s$.
(c) $\sum_{n=1}^{\infty} f(m,n)$ and $\sum_{m=1}^{\infty} f(m,n)$ both converge absolutely.
(d) If $Am=\sum_{n=1}^{\infty} f(m,n)$ and $B_n=\sum_{m=1}^{\infty} f(m,n)$, both series $\sum Am$ and $\sum B_n$ coverge absolutely and both have sum s, That is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n)=s.$$

21. State and prove mertens Theorem.
22. Assume that $\sum_{n=0}^{\infty} a_n$ converges absolutely and has sum and suppose $\sum_{n=0}^{\infty} b_n$ converges with sum B. Then the caucy product of these two series converges and has sum AB.
23. If a series is converges with sum s, then it is also (c,1) summabke with cesaro sum s.
24. Assume that $\sum a_n(z-z_o)^n$ converges if $z\epsilon B(z_o;r)$. suppose that the equation $f(z)=\sum_{n=0}^{\infty} a_n(z-z_o)^n$. Then for each point z_1 in s, there exists a neighbourhood $B(z_1;R)\subseteq s$ in which f as a power series expansion of the form
$f(z)=\sum_{k=0}^{\infty} b_k(z-z_1)^k$, where $b_k=\sum_{n=k}^{\infty}\binom{n}{k}a_n(z_1-z_0)^{n-k}$,(k=0,1,2,…..)
25. Assume that $\sum a_n(z-z_0)^n$ converges for each z in $B(z_0;r)$. Then the function f defined by the equation $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$, if $z\epsilon B(z_o;r)$ has a derivative $f^1(z)$ for each z in $B(z_0;r)$, given by $f'(z)=\sum_{n=1}^{\infty} n a_n(z-z_0)^{n-1}$.
26. Sate and prove Bernstein Teorem.
27. Assume f and all its derivative are non negative on a compact interval [b,b+r]. Then, if $b\leq x<b+r$, the Taylor's series $\sum_{k=0}^{\infty}\frac{fk(b)}{k!}(x-b)^k$, converges to f(x).
28. State and prove Tauber's Theorem.
29. Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ for $-1<x<1$, and assume that $\lim_{n\to\infty} n a_n=0$. If $f(x)\to s$,as $x\to 1-$, then $\sum_{n=0}^{\infty} a_n$ converges and has sum s.

UNIT-V

30. Let $\{f_n\}$ be a sequence of function defined on a set s. There exists a function f such that $f_n\to f$ uniformly on s, iff the following condition is satisfied: for every $\varepsilon>0$ there exists on N such that m>N and n>N implies $|f_m(x)-f_n(x)|<\varepsilon$, for every x in s.
31. Let $\propto$ be of bounded variation on [a,b]. Assume that each term of the sequence $\{f_n\}$ is a real valued function such that $f_n\epsilon R(\propto)$ on [a,b] for each n=1,2,.. Assume that $f_n\to f$ uniformly on [a,b] and define $g_n(x)=\int_a^x f_n(t)d\propto(t)$ if $x\epsilon[a,b]$, n=1,2,3,…. Then we have;
    (a) $f\epsilon R(\propto)$ on [a,b].
    (b) $g_n\to g$ uniformly on [a,b], where $(x)=\int_a^x f(t)d\propto(t)$.
32. Let $\{f_n\}$ be a boundedly convergent sequence on[a,b]. Assume that each $f_n\epsilon R$ on [a,b], and that the limit function $f\epsilon R$ on [a,b]. Assume also that there is a partition p of [a,b] say $p=\{x_o,x_1,x_2,…..\}$ such that, on every sub interval [c,d] not containing any of the points x_k, the sequence $\{f_n\}$ converges uniformly to f, Then we have $\lim_{n\to\infty}\int_a^b f_n(t)\,dt=\int_a^b \lim_{n\to\infty} f_n(t)\,dt=\int_a^b f(t)dt$.
33. Assume that each term of $\{f_n\}$ is a real valued function having a finite derivation at each point of a_n open interval (a,b). Assume that for at least one point x_o in (a,b)

the sequence $\{f_n(x_0)\}$ converges. Assume further that there exists a function g such that $f_n' \to g$ uniformly on (a,b).

(a) *There exists a function f such that $f_n \to f$ uniformly on (a,b).*

(b) *For each x in (a,b) the derivative f'(x) exists and equals (x).*

34. *State and prove Dirichlet's test for uniform convergence.*

Let $F_n(x)$ denote the n^{th} partial sum of the series $\sum f_n(x)$ where each f_n is a complex valued function defined on a set s. Assume that $\{f_n\}$ is uniformly bounded on s. Let $\{g_n\}$ be a sequence of real valued *functions such that $g_{n+1}(x) \le g_n(x)$ for each x in s and for every n=1,2,.. and assume that $g_n \to 0$ uniformly on s. Then the series $\sum f_n(x) g_n(x)$ c*