D.K.M COLLEGE FOR WOMEN (AUTONOMOUS),VELLORE-1

DEPARTMENT OF MATHEMATICS

CLASS : II M.SC(MATHEMATICS)

SUBJECT : TOPOLOGY **SUB.CODE :15CPMA3C**

UNIT – I SECTION-A 6 MARKS

1. Let X be a set; Let $\mathfrak{B}$ be a basis for a topology J on X. Then J equals the collection of all unions of elements of $\mathfrak{B}$.

2. Let X be a topological space, Supppose that e is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C of e. Such that x ϵ c $\subset$ U. Then e is a basis for the topology of X

3. The topologies of $\mathfrak{R}_\ell$ and $\mathfrak{R}_k$ are strickly finer than the standard topology on $\mathfrak{R}$, but are not comparable with one another.

4. Let X be a topological space; let A be a subset of X. Suppose that for each xϵA there is an open set U containing x such that U $\subset$ A. Show that A is open in x.

5. If $\mathfrak{B}$ is a basis for the topology of X and e is a basis for the topology of Y, then the collection D = { B x C / B ϵ $\mathfrak{B}$ and C ϵ e } is a basis for the topology of X x Y.

6. The collection δ = { π_1^{-1} (u) / U open in X } $\cup$ { π_2^{-1} (u) / open in X } is a subbasis for the product topology on X x Y.

7. If $\mathfrak{B}$ is a basis for the topology of X then the collection $\mathfrak{B}_y$ ={ B $\cap$ y/ B$\epsilon\mathfrak{B}$ } is a basis for the subspace topology on X.

8. Let Y be a subspace of X. If U is open in Y and y is open in X, then U is open in X.

9. Let Y be a subspace of X. If A is closed in Y and y is closed in X, then A is closed in X.

10. Let Y be a subspace of x, let A be a subset of Y; Let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A}$ $\cap$ Y.

11. Let A be a subset of the topological space X; Let A^1 be the set of all limit points of A. Then $\bar{A} = A \cup A^1$.

12. A subset of a topological space is closed if and only if it contains all its limit points.

13. Every finite point set in a Hausdroff space X is closed.

14. Let X be a space satisfying the T_1 axiom; Let A be a subset of X. Then the point x is a limit point of A^1 if and only if every neighbourhood of x contains infinitely many points of A.

15. If X is a Hausdroff space, then a sequence of points of X converges to atmost one point of X.

UNIT-II

1. State and prove the posting lemma.

2. Let f : A → X x Y be given by the equation.
f (a) = (f_1(a), f_2(a)). Then f is continuous if and only if the functions f_1 : A $\to X$ *and* f_2 : A →Y are continuous. The maps f_1 and f_2 are called the coordinate functions of f.

3. State and Prove maps into products.

4. State and prove comparison of the box and product topologies.

5. Suppose the topology on each space X_α is given by a basis $\mathfrak{P}_\alpha$. The collection of all sets of the form $\Pi_{\alpha\epsilon J}\ \mathfrak{B}_\alpha$ where $B_\alpha\ \epsilon\,\mathfrak{B}_\alpha$ for each α, will serve as a basis for the box topology on $\Pi_{\alpha\epsilon J}\ X_\alpha$.
6. Let A_α be a subspace of X_α, for each $\alpha \in J$. Then πX_α is a subspace of πX_α if both products are given the box topology, or if both products are given the product topology.

7. If each space X_α is a hausdroff space, then πX_α is a Housdroff space in both the box and product topologies.

8. Let { X_α } be an indexed family of spaces, Let $A_\alpha \subset X_\alpha$ for each α. If is πX_α is given either the product or the box topology, then $\pi A_{\alpha=\overline{\pi A_\alpha}}$

9. Let X be a matric space with metric d, define $\bar{d}: X\,x\,X \to \mathfrak{R}$ by the equation $\bar{d}(\,x, y\,)$= min { d(x, y), 1 }. Then $\bar{d}$ is a metric that induces the same topology as d.

10. Let d^* and d^1 be two metrices on the set X; Let J and J^1 be the topologies they induces respectively. Then J^1 is finer then J if and only if for each x in X and each $\Sigma > 0$, there exists a $\delta > 0$ such that B d^1 (x, δ) $\subset B\ d\ (x, \varepsilon)$.

11. The uniform topology on $\Re^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different if J is finite.

12. State and prove the sequence lemma.

13. If X is a topological space, and if f, g : X $\longrightarrow R$ are continuous functions, then f + g, f – g, and fog are continuous . If g(x) $\neq$ 0 for all x, then f / g is continuous.

UNIT-III

1. If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty set A and B whose union is Y, neither of which contains a limit point of the other, the space Y is connected if there exists no separation of Y.
2. If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

3. The union of a collection of connected subspaces of X that have a point in common is connected.

4. Let A be a connected subspaces of X. If A $\subset B \in \bar{A}$, then B is also connected.

5. The components of x are connected disjoint subspaces of X whoses union is X, such that each nonempty connected subspace of x intersects only one of them.

6. A space X is locally connected if and only if for every open set U of x, each component of U is open in X.

7. A space X is locally path connected if and only if for every open set U of x, each path component of U is open in X.

8. If A is connected and if B is a set obtained from A by adjoining (or) all limits points of A. Then B is connected.

9. A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

UNIT-IV

1. Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

2. Every closed subspace of a compact space is compact.

3. Every compact subspace of a housdroff space is closed.
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4. Let f : X → Y be a bijective continuous function. If X is compact and Y is Housdroff, then f is a homomorphism.
5. Every closed interval in $\Re$ is compact.

6. A subspace A of $\Re^n$ is compact if and only if it is closed and is bounded in the Euclidean metric d or the square metric ρ.

7. State abd Prove : Extreme value theorem.

8. Let f : X → Y be continuous, where Y is an ordered set in the order topology. If X is compact, then there exixts points C and d in X such that f(c) $\leq f(x) \leq f(d)\ for\ every\ x\ \in X$.

9. State and prove uniform continuity theorem.

10. Let X be a nonempty compact Housdroff space. If X has no isolated points, then X is uncountable.

11. Let f : X → Y be a continuous map of the compact metric space (X, dx) to the metric space (y, d_y). Then f is uniformly continuous.

12. Let X be locally compact Housdroff; Let A be a subspace of X. If A is closed in X or open in X, then A is locally compact.

13. Let X be a Housdroff Space. Then X is locally compact if and only if given x in X, and given a neighbourhood U of x, there is a neighbourhood v of x such that $\bar{v}$ is compact and $\bar{v}$ ⊂ U.

14. A space X is homeomorphic to an open subspace of a compact Housdroff space if and only if X is locally compact Housdroff.

UNIT-V

1. Let X be a topological space.
   (a) Let A be a subset of X. If there is a sequence of points of A converng to x, then x∈ $\bar{A}$; the converse holds if X is first-countable.
   (b) Let f : X → Y. If f is continous, then for every converget sequence $X_n \to x$ in x, the sequence f(x_n) converges to f (x). The converse holds if X is first countable.

2. Suppose that X has a countable basis. Then
   (a) Every open covering on X contains a countable subcollection covering X.
   (b) There exists a countable subset of X that is dense in X.

3. Let X be a topological space, Let one-point sets in X be closed.
   (a) X be regular if and only if given a point x of X and a neighbourhood U of x, there is a neighbourhood v of x such that $\bar{v} \subset$ c.
   (b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set v containing A such that $\bar{v} \subset$ c.

4. Every regular space with a countable basis is normal.

5. Every metrizable space is normal.

6. Every compact Housdroft space is normal.

7. Every well-ordered set X is normal in the order topology.

8. A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

UNIT-I SECTION-B 15 MARKS

1. Let B and B^1 be bases for the topology J and J^1 respectively, on X. Then the following are equivalent,
   (1) $\mathfrak{I}^1$ is finer than J.
   (2) For each x ∈ X and each basis element B ∈ $\mathfrak{B}$ containing x , there is a basis element B ∈ $\mathfrak{B}$ such that $x \in B^1 \subset B$.

2. If A is a subspace of x and B is a subspace of Y, then the product topology on A x B is the same as the topology A x B inherits as a subspace of X x Y.

3. Let X be an ordered set in the order topology; Let Y be a subset of X that is convex in X, Then the order topology on Y is the same as the topology Y inherits as a subspace of x.

4. Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

5. Let A be a subset of the topological space X.
   (a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

(b) Suppose the topology of x is given by a basis, then x ∈ $\bar{A}$ if and only if every basis element B containing x intersects A.

UNIT-II

1. Let X and Y be topological spaces; Let f : X→ Y Then the following are equivalent;
   (1) f is continuous
   (2) for every subset A of x, on has f($\bar{A}$) ⊂ $\overline{f(A)}$.
   (3) For every closed set B of Y, the set f^{-1} (B) is closed in X.
   (4) For each x ∈ X and each neighbourhood v of f(x), there is a neighbourhood U of x such that f (U) ⊂ v.
   If the contion in (4) holds for the points x of X, we say that f is continuous at the point x.

2. State and Prove Rules for constructing continuous mapping.

3. Let X, Y and Z be topological spaces,
   (a) Constant function : If f : X ⟶ Y maps all of X into the single point of Y, then f is continuous.
   (b) Inclusion : If A is a subspace of X, the inclsion function j : A ⟶ X is continuous.
   (c) Composites : If f : X ⟶ Y and g : Y ⟶ Z are continuous, then the map g o f : X ⟶ Y is continuous.
   (d) Restricting the domain: If f : X ⟶ Y is continuous, and if A is a subspace of x, then the restricted function f / A : A ⟶ Y is continuous.
   (e) Local formulation of continunity: The map f : X ⟶ Y is continous if x can be written as the union of opensets U_α such that f / U_α is continuous U_α each α .

4. Let f : A → $\pi_{\alpha\epsilon J}\, X_\alpha$ be given by the equation f(a) = (f_a (a) $)_{\alpha\epsilon J}$, *where* $f_\alpha\!: A \to X_\alpha$ for each α. Let πX_α have the product topology, then the function f is continous if and only of each function ρ_0 is continuous.

5. The topologies on $\mathfrak{R}^n$ induced by the Euclidean matrix d and the square matrc S are the same as the product topology on $\mathfrak{R}^n$.

6. Let $\bar{d}\,(a, b) = \min\{\,|a - b|n\,\}$ be the standard bounded metrix on $\mathfrak{R}$. If X and Y are two points of $\mathfrak{R}^m$, define D(x, y) = sup{ $\frac{\overline{d\ (xi,yi\ )}}{i}$ }. Then D is a metric that induces the product topology on $\mathfrak{R}^w$.

7. Let f : X $\to$ Y. Let X and Y be metrizable with metrices dx and dy, respectively. Then continunity of f is equivalent to the requirement that given x $\in X$ and given $\varepsilon < 0$ there exists $\delta > 0$ such that $d_x(x, y) < \delta \Rightarrow d_y(f(x), f(y)) < \varepsilon$.

8. Let f : X $\to$ Y. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to f(x). The converse holds if x is metrizable.

UNIT-III

1. A finite Cartesian prodruct of connected space is connected.

2. If L is a linear continum in the order topology, then L is connected, and so are intervals and rays in L.

3. State and Prove: " Intermediate value theorem".

4. Let f : X $\to$ Y be a continuous map, where x is a connected space and y is an ordered set in the order topology. If a and b are two points of x and if r is a point of y lying between f(a) and f(b), then there exists a point C of x such that f(c) = r.

5. If path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path connected subspaces of X intersects only one of them.

UNIT-IV

1. If x is a topological space, each path component of x lies in a component of x, If x is locally path connected, then the components and the path components of x are the same.

2. Every compact subspace of a housdraff space is closed

3. The image of a compact space under a continuous map is compact.

4. The product of finitely many compact space is compact.

5. Consider the product space X x Y, where Y is compact. If N is an open set of X x Y containing the slice x_0 x y, W is a neighbourhood of x_0 in x.

6. State and Prove: "The tube lemma"

7. Let X be a topological space, Then x is compact if and only if for every collection e of closed sets in X having the finite intersection property, the intersection $\cap_c \in c^e$ of all the elements of e is nonempty.

8. Let X be a simply ordered set having the least upper bounded property. In the order topology, each closed interval in x is compact

9. Let A be an open covering of the matric space (x, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there esists an element of A containing it. The number δ is called a lebesque number for the covering A.

10. Let f : X $\rightarrow$ Y be a continuous map of the compact metric space (x, d_x) to the metric space (y, d_y). Then f is uniformly continuous.

11. Compactness implies limit point compactness, but nor conversely.

12. Let X be a metrizable space, then the following are equivalent;
    (1) X is compact
    (2) X is limit point compact
    (3) X is sequentially compact

UNIT-V

1. State and prove " Urysohn lemma"

2. Let X be a normal space; Let A and B be disjoint closed subsets of X. Let [a, b] be a closed interval in the real line. Then there exists a continuous map f : X $\rightarrow$ [a, b] such that f(x) = b for every x in B.

3. State and Prove : Urysohn metrization Theorem.

4. Every regular space X with a countable basis is metrizable.

5. State and Prove: Imbedding Theorem
6. State and Prove: Tietze Extension Theorem

7. Let X be a normal space; Let A be a closed subspace of X.
    (a) Any continuous map of A into the closed interval [a, b] of $\Re$ may be extended to a continuous map of all of x into [a, b].
    (b) Any continous map of A into $\Re$ may be extended to a continuous map of all of x into $\Re$.