ALGEBRA

Unit-I : Theory of Equations

Polynomial Equations :

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n \text{, where}$$

n is a positive integer and $a_0, a_1, a_2 \dots a_n$ are constants is called an algebric equation or a Polynomial equation of the n^{th} degree, if $a_0 \neq 0$.

Theorem :

1. Every Polynomial equation of the n^{th} degree has n roots and only n roots.
2. In a polynomial equation with real coefficients, imaginary roots occur in conjugate pairs.
3. If $f(x)=0$ be a polynomial equation with rational coefficients then irrational roots occur in pairs.

Problems

1. Solve $x^4+4x^3+5x^2+2x-2=0$, given $-1+i$ is a root.

Soln:

Since $-1+i$ is a root, $-1-i$ is also a root.

The real factor corresponding to the two roots is

$$[x-(-1+i)][x-(-1+i)]$$

$$= [(x+1)-i][(x+1)+i]$$

$$= (x+1)^2+1$$

$$= x^2+2x+2$$

$$
\begin{array}{r|l}
 & x^2+2x-1 \\
\hline
x^2+2x+2 & \cancel{x^4}+4x^3+5x^2+2x-2 \\
 & \cancel{x^4}+2x^3+2x^2 \\
 & (-)\quad (-) \quad (-) \\
\hline
 & \cancel{2x^3}+3x^2+2x \\
 & \cancel{2x^3}+4x^2+4x \\
 & (-)\quad (-) \quad (-) \\
\hline
 & -x^2-2x-2 \\
 & -x^2-2x-2 \\
 & (+)\quad (+) \quad (+) \\
\hline
 & 0
\end{array}
$$

The other roots are given by $x^2+2x-1=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4-4(-1)}}{2\cdot 1} = \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= -1 \pm \sqrt{2}$$

$\therefore$ The roots are $-1+i, -1-i, -1+\sqrt{2}, -1-\sqrt{2}$

(2) Solve $x^4-10x^3+26x^2-10x+1=0$, given that $2+\sqrt{3}$ is a root of the equation.

Soln :

Since $2+\sqrt{3}$ is a root, $2-\sqrt{3}$ is also a root.

The real factor corresponding to the two roots is

$$[x-(2+\sqrt{3})][x-(2-\sqrt{3})]$$

$$= [(x-2)-\sqrt{3}][(x-2)+\sqrt{3}]$$

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$= (x-2)^2 - (\sqrt{3})^2$

$= x^2 - 4x + 4 - 3$

$= x^2 - 4x + 1.$

$$
\begin{array}{r|l}
 & x^2 - 6x + 1 \\
\hline
x^2-4x+1 & \cancel{x^4} - 10x^3 + 26x^2 - 10x + 1 \\
 & \cancel{x^4} - 4x^3 + x^2 \\
 & (-) \quad (+) \quad (-) \\
\hline
 & -\cancel{6x^3} + 25x^2 - 10x \\
 & -\cancel{6x^3} + 24x^2 - 6x \\
 & (+) \quad (-) \quad (+) \\
\hline
 & x^2 - 4x + 1 \\
 & x^2 - 4x + 1 \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 0
\end{array}
$$

The other roots are given by $x^2 - 6x + 1 = 0$

$x = \dfrac{6 \pm \sqrt{36 - 4\cdot1\cdot1}}{2\cdot1} = \dfrac{6 \pm \sqrt{32}}{2}$

$= \dfrac{6 \pm 4\sqrt{2}}{2}$

$= 3 \pm 2\sqrt{2}$

$\therefore$ The roots are $2+\sqrt{3}$, $2-\sqrt{3}$, $3+2\sqrt{2}$, $3-2\sqrt{2}$

(3) One of the roots of the equation $3x^5 - 4x^4 - 42x^3 + 56x^2 + 27x - 36 = 0$ is $\sqrt{2}+\sqrt{5}$. Find the other roots.

Soln: $\sqrt{2}+\sqrt{5}$ is a root.

$\therefore \sqrt{2}-\sqrt{5}$, $-\sqrt{2}+\sqrt{5}$, $-\sqrt{2}-\sqrt{5}$ are also roots.

The real factors corresponding to these roots is

$[x-(\sqrt{2}-\sqrt{5})][x-(-\sqrt{2}+\sqrt{5})][x-(-\sqrt{2}-\sqrt{5})][x-(\sqrt{2}+\sqrt{5})]$

$$= [(x-\sqrt{2})+\sqrt{5}][(x+\sqrt{2})-\sqrt{5}][(x+\sqrt{2})+\sqrt{5}][(x-\sqrt{2})-\sqrt{5}]$$

$$= [(x-\sqrt{2})^2-(\sqrt{5})^2][(x+\sqrt{2})^2-(\sqrt{5})^2]$$

$$= [x^2-2\sqrt{2}x+2-5][x^2+2\sqrt{2}x+2-5]$$

$$= [x^2-2\sqrt{2}x-3][x^2+2\sqrt{2}x-3]$$

$$= [(x^2-3)-2\sqrt{2}x][(x^2-3)+2\sqrt{2}x]$$

$$= (x^2-3)^2-(2\sqrt{2}x)^2$$

$$= x^4-6x^2+9-8x^2$$

$$= x^4-14x^2+9$$

$$\begin{array}{r|l}
 & 3x-4 \\
\hline
x^4+0x^3-14x^2+0x+9 & \cancel{3x^5}-4x^4-42x^3+56x^2+27x-36 \\
 & \cancel{3x^5}+0-42x^3+0+27x \\
 & (-)\quad(-)\quad(+)\quad(-)\quad(-) \\
\hline
 & -4x^4+56x^2-36 \\
 & -4x^4+56x^2-36 \\
 & (+)\quad(-)\quad(+) \\
\hline
 & 0
\end{array}$$

The other root is $3x-4=0$

$$\boxed{x=4/3}$$

The five roots are $\sqrt{2}+\sqrt{5}$, $\sqrt{2}-\sqrt{5}$, $-\sqrt{2}+\sqrt{5}$, $-\sqrt{2}-\sqrt{5}$, $4/3$

Relations between the roots and coefficients of equations

Let the equation be

$$x^n+P_1x^{n-1}+P_2x^{n-2}+\dots+P_{n-1}x+P_n=0$$

Let the roots of this equation be

$\alpha_1, \alpha_2, \dots \alpha_n$.

Then $x^n + P_1 x^{n-1} + P_2 x^{n-2} + \dots + P_n$

$= (x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$

$= x^n - \Sigma\alpha_1 x^{n-1} + \Sigma\alpha_1\alpha_2 x^{n-2} - \dots$

$+ (-1)^n \alpha_1\alpha_2 \dots \alpha_n$

$= x^n - S_1 x^{n-1} + S_2 x^{n-2} - \dots + (-1)^n S_n$.

Where S_r is the sum of the products of the roots $\alpha_1, \alpha_2 \dots \alpha_n$ taken r at a time.

Equating the co-efficients on both sides, we get $P_1 = -S_1$

$P_2 = S_2$

$P_3 = -S_3$

$\vdots$

$P_n = (-1)^n S_n$

$\Rightarrow -P_1 = S_1 =$ Sum of the roots

$P_2 = S_2 =$ Sum of the product of the roots taken 2 at a time.

$-P_3 = S_3 =$ Sum of the product of the roots taken 3 at a time.

$\vdots$

$(-1)^n P_n = S_n =$ Product of the roots.

<u>Note:</u>

Let the equation be $ax^3 + bx^2 + cx + d = 0$

Let the roots of this equation be α, β, γ, then

$S_1 : \alpha + \beta + \gamma = -b/a$

$S_2 : \alpha\beta + \beta\gamma + \gamma\alpha = c/a$

$S_3 : \alpha\beta\gamma = -d/a$

Symmetric Functions of the roots:

(1) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$. Find the value of

i) $\sum\alpha^2$ ii) $\sum\alpha^2\beta$ iii) $\sum\alpha^2\beta^2$ iv) $\sum 1/\alpha^2$

v) $\sum\alpha^3$ vi) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

Soln:

$S_1 : \sum\alpha = -p$

$S_2 : \sum\alpha\beta = q$

$S_3 : \alpha\beta\gamma = -r$

i) $\sum\alpha^2 = \alpha^2+\beta^2+\gamma^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= (-p)^2 - 2q = p^2 - 2q$

ii) $\sum\alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$

$= (\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 3\alpha\beta\gamma$

$= (-p)q - 3(-r)$

$= 3r - pq$

iii) $\sum\alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

$= (\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= q^2 - 2(-r)(-p)$

$= q^2 - 2pr.$

iv) $\sum 1/\alpha^2 = 1/\alpha^2 + 1/\beta^2 + 1/\gamma^2$

$= \dfrac{\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2}{\alpha^2\beta^2\gamma^2} = \dfrac{q^2-2pr}{(-r)^2}$

$= \dfrac{q^2-2pr}{r^2}$

$$\Sigma\alpha(\beta^2+\gamma^2) = \alpha(\beta^2+\gamma^2)+\beta(\alpha^2+\gamma^2)+\gamma(\alpha^2+\beta^2)$$
$$= (\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) - 3\alpha\beta\gamma$$
$$= (-p)q - 3(-r)$$
$$= 3r - pq.$$

v) $$\Sigma\alpha^3 = \alpha^3+\beta^3+\gamma^3$$
$$= (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2) - \Sigma\alpha(\beta^2+\gamma^2)$$
$$= (-p)(p^2-2q) - (3r-pq)$$
$$= -p^3+2pq-3r+pq$$
$$= -p^3+3pq-3r$$

vi) $$(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$$
$$= (\alpha+\beta+\gamma-\gamma)(\beta+\gamma+\alpha-\alpha)(\gamma+\alpha+\beta-\beta)$$
$$= (-p-\gamma)(-p-\alpha)(-p-\beta)$$
$$= -(p+\gamma)(p+\alpha)(p+\beta)$$
$$= -[p^3+p^2(\alpha+\beta+\gamma)+p(\alpha\beta+\beta\gamma+\gamma\alpha)+\alpha\beta\gamma]$$
$$= -[p^3+p^2(-p)+p(q)+(-r)]$$
$$= r-pq.$$

(2) If $\alpha,\beta,\gamma,\delta$ be the roots of the biquadratic equation $x^4+px^3+qx^2+rx+s=0$. Find

i) $\Sigma\alpha^2$ ii) $\Sigma\alpha^2\beta\gamma$ iii) $\Sigma\alpha^2\beta^2$ iv) $\Sigma\alpha^3\beta$ v) $\Sigma\alpha^4$

Soln: S_1 : $\alpha+\beta+\gamma+\delta$

S_2 : $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = q$

S_3 : $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -r$

S_4 : $\alpha\beta\gamma\delta = s$

i) $\sum\alpha^2 = \alpha^2+\beta^2+\gamma^2+\delta^2$

$= (\alpha+\beta+\gamma+\delta)^2 - 2\sum\alpha\beta$

$= (-p)^2 - 2q$

$= p^2 - 2q$

ii) $\sum\alpha^2\beta\gamma = (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)(\alpha+\beta+\gamma+\delta)$
$- 4\alpha\beta\gamma\delta$

$= (\sum\alpha\beta\gamma)(\sum\alpha) - 4\alpha\beta\gamma\delta$

$= (-r)(-p) - 4s$

$= pr - 4s$

iii) $\sum\alpha^2\beta^2 = \alpha^2\beta^2+\alpha^2\gamma^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2$

$= (\sum\alpha\beta)^2 - 2\sum\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$

$= q^2 - 2(pr-4s) - 6s$

$= q^2 - 2pr + 2s$

iv) $\sum\alpha^3\beta = (\sum\alpha^2)(\sum\alpha\beta) - \sum\alpha^2\beta\gamma$

$= (p^2-2q)q - (pr-4s)$ (using ii)

$= p^2q - 2q^2 - pr + 4s$

v) $\sum\alpha^4 = (\sum\alpha^2)^2 - 2\sum\alpha^2\beta^2$

$= (p^2-2q)^2 - 2(q^2-2pr+2s)$

$= p^4 - 4p^2q + 2q^2 + 4pr - 4s$

Note :

If α, β, γ are the roots of an equation $x^3+px^2+qx+r=0$

* If the roots are in A.P then the roots are in the form $a-d, a, a+d$.

* If the roots are in G.P then the roots are in the form $\frac{K}{r}, K, Kr$.

* If the roots are in H.P then the roots will satisfy $\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma}$

If the roots of the equation $x^3+px^2+qx+r=$ are in A.P. Prove that $2p^3 - 9pq + 27r = 0$. Show that the above condition is satisfied by the equation $x^3 - 6x^2 + 13x - 10 = 0$. Hence solve the equation.

Soln:

Let the roots of the equation

$x^3 + px^2 + qx + r = 0$ — (1) be $a-d, a, a+d$

S_1 : $a-d + a + a+d = -p$ — (2)

S_2 : $a(a-d) + a(a+d) + (a+d)(a-d) = q$ — (3)

S_3 : $a(a-d)(a+d) = -r$ — (4)

(2) $\Rightarrow$ $3a = -p \Rightarrow \boxed{a = -p/3}$

(3) $\Rightarrow$ $a^2 - \cancel{ad} + a^2 + \cancel{ad} + a^2 - d^2 = q$

$\Rightarrow$ $3a^2 - d^2 = q$

$3\left(\frac{-p}{3}\right)^2 - d^2 = q$

$d^2 = p^2/3 - q$

(4) $\Rightarrow$ $a(a^2 - d^2) = -r$

$-\frac{p}{3}\left[\frac{p^2}{9} - \left(\frac{p^2}{3} - q\right)\right] = -r$

$-\frac{p}{3}\left[\frac{p^2 - 3p^2 + 9q}{9}\right] = -r$

$\Rightarrow p[-2p^2 + 9q] = 27r$

$-2p^3 + 9pq - 27r = 0$

$\Rightarrow 2p^3 - 9pq + 27r = 0$ — (5)

Given equation is $x^3 - 6x^2 + 13x - 10 = 0$

$p = -6 \quad q = 13 \quad r = -10$

Sub. in (5)

$2(-6)^3 - 9(-6)(13) + 27(-10)$

$= -432 + 702 - 270$

$= 0$

$\therefore$ The condition is satisfied.

The roots are in A.P

S_1 : $3a = -p$

$3a = 6 \Rightarrow \boxed{a = 2}$

S_2 : $3a^2 - d^2 = q$

$3(4) - d^2 = 13 \Rightarrow -d^2 = 13 - 12 = 1$

$d^2 = -1$

$d = \pm i$

$\therefore$ The roots are $2-i, 2, 2+i$.

(2) Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$ whose roots are in G.P.

Soln: Let the roots be $\frac{k}{r}, k, kr$.

S_1 : $\frac{k}{r} + k + kr = -42/27$ — (1)

S_2 : $\frac{k^2}{r} + k^2 r + k^2 = -28/27$ — (2)

S_3 : $\frac{k}{r} \cdot k \cdot kr = -(-8)/27$

$k^3 = 8/27 = \frac{2^3}{3^3}$

$\boxed{K = 2/3}$

Sub $K = 2/3$ in ①

① $\Rightarrow K\left(\frac{1}{r}+1+r\right) = -42/27$

$$\frac{2}{3}\left(\frac{1+r+r^2}{r}\right) = -\frac{42}{27}$$

$$\frac{1+r+r^2}{r} = \frac{-7}{3}$$

$$3+3r+3r^2 = -7r$$

$$\Rightarrow 3r^2+10r+3 = 0$$

$$(3r+1)(r+3) = 0$$

$$r = -3, -1/3$$

$\therefore$ The roots are $-\frac{2}{9}, \frac{2}{3}, -2$.

③ Solve $6x^3-11x^2+6x-1=0$ given that the roots are in H.P.

Soln: $6x^3-11x^2+6x-1=0$ — ①

Put $x = 1/y$ in ①

$$6(1/y)^3 - 11(1/y)^2 + 6(1/y) - 1 = 0$$

$$\Rightarrow 6-11y+6y^2-y^3 = 0$$

$$y^3-6y^2+11y-6 = 0 \text{ — ②}$$

The roots of ② is in A.P

Let the roots be $a+d$, a, $a-d$.

S_1 : $a-\cancel{d}+a+a+\cancel{d} = -(-6)$

$$3a = 6 \Rightarrow a = 2$$

S_2 : $a^2-\cancel{ad}+a^2+\cancel{ad}+a^2-d^2 = 11$

$$3a^2-d^2 = 11 \text{ — ③}$$

Sub $a = 2$ in (3)

$3(2)^2 - d^2 = 11$

$12 - d^2 = 11 \Rightarrow -d^2 = 11 - 12 = -1$

$d^2 = 1 \Rightarrow d = \pm 1$

The roots of (2) are $1, 2, 3$

The roots of (1) are $1, 1/2, 1/3$.

Transformation of Equation

Diminish or Increase the roots of the given equation by the given quantity.

(1) Diminish by 3, the roots of $x^4 + 3x^3 - 2x^2 - 4x - 3 = 0$

$$
\begin{array}{c|ccccc}
3 & 1 & 3 & -2 & -4 & -3 \\
 & 0 & 3 & 18 & 48 & 132 \\
\hline
 & 1 & 6 & 16 & 44 & \boxed{129} \\
 & 0 & 3 & 27 & 129 & \\
\hline
 & 1 & 9 & 43 & \boxed{173} & \\
 & 0 & 3 & 36 & & \\
\hline
 & 1 & 12 & \boxed{79} & & \\
 & 0 & 3 & & & \\
\hline
 & 1 & \boxed{15} & & &
\end{array}
$$

$\therefore$ The required equation is

$x^4 + 15x^3 + 79x^2 + 173x + 129$.

(2) Increase by 2, the roots of $x^4 - x^3 - 10x^2 + 4x + 24 = 0$ and hence solve the equation.

-2	1	-1	-10	4	24
	0	-2	6	8	-24
	1	-3	-4	12	$\boxed{0}$
	0	-2	10	-12	
	1	-5	6	$\boxed{0}$	
	0	-2	14		
	1	-7	$\boxed{20}$		
	0	-2			
	1	$\boxed{-9}$			

$\therefore$ The required equation is

$$x^4 - 9x^3 + 20x^2 = 0$$

$$x^2(x^2 - 9x + 20) = 0$$

$$x^2(x-4)(x-5) = 0$$

$$\Rightarrow \quad x = 0, 0, 4, 5.$$

$\therefore$ The roots of (2) are $0, 0, 4, 5$.

$\therefore$ The roots of (1) are $-2, -2, 2, 3$.

Note:

1. Suppose $\alpha_1, \alpha_2 \dots \alpha_n$ are the roots of $f(x) = 0$. Diminishing the roots of an equation by 'h', the required roots are $\alpha_1 - h, \alpha_2 - h, \dots \alpha_n - h$.

2. If the roots of the given equation are diminished by 'h' where $h = \dfrac{-a_1}{na_0} = \dfrac{\text{sum of the roots}}{\text{degree}}$, then the second term of the resulting equation will be absent.

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3. To increase the roots by 'h', decrease the roots by '-h'.

1. Transform the equation $x^4 - 8x^3 - x^2 + 68x + 60 = 0$ into one which does not contain the term in x^3. Hence solve the equation.

$$x^4 - 8x^3 - x^2 + 68x + 60 = 0 \quad \text{— (1)}$$

$$h = \frac{-a_1}{na_0} = \frac{-(-8)}{4(1)} = \frac{8}{4} = 2.$$

$$\begin{array}{c|ccccc}
2 & 1 & -8 & -1 & 68 & 60 \\
 & 0 & 2 & -12 & -26 & 84 \\
\hline
 & 1 & -6 & -13 & 42 & \boxed{144} \\
 & 0 & 2 & -8 & -42 & \\
\hline
 & 1 & -4 & -21 & \boxed{0} & \\
 & 0 & 2 & -4 & & \\
\hline
 & 1 & -2 & \boxed{-25} & & \\
 & 0 & 2 & & & \\
\hline
 & 1 & \boxed{0} & & &
\end{array}$$

∴ The required equation is

$$x^4 - 25x^2 + 144 = 0$$

Let $x^2 = y$

$$y^2 - 25y + 144 = 0$$

$$(y-9)(y-16) = 0$$

$$y = 9, 16$$

$$x = \pm 3, \pm 4.$$

144

-9 -16

-25

∴ The roots of eqn. (2) are $-3, -4, 3, 4$

∴ The roots of eqn. (1) are $-1, -2, 5, 6$.

Reciprocal Equation:

An equation $f(x)=0$ is called a reciprocal equation if $f(x) = f(1/x)$.

i) A reciprocal equation of odd degree with like signs has one of its roots equal to -1. Hence $x+1$ is a factor of $f(x)$.

ii) A reciprocal equation of odd degree with unlike sign has one of its roots equal to $+1$. Hence $x-1$ is a factor of $f(x)$.

iii) A reciprocal equation of even degree with unlike signs with middle term absent has two of its roots equal to $+1$ and -1. Hence (x^2-1) is a factor of $f(x)$.

iv) A reciprocal equation of even degree (or) unlike signs with the presence of middle term then solve the equation.

① Solve $x^5+4x^4+x^3+x^2+4x+1=0$

Soln

$f(x)$: $x^5+4x^4+x^3+x^2+4x+1=0$ —— ①

$f(1/x)$: $\frac{1}{x^5}+\frac{4}{x^4}+\frac{1}{x^3}+\frac{1}{x^2}+\frac{4}{x}+1=0$

$\Rightarrow 1+4x+x^2+x^3+4x^4+x^5=0$

$f(x)=f(1/x)$

$f(x)$ is a reciprocal equation.

This is a reciprocal equation of odd degree with like signs.

$\therefore$ $x=-1$ is a root of ①

$(x+1)$ is a factor of ①

-1	1	4	1	1	4	1
	0	-1	-3	2	-3	-1
	1	3	-2	3	1	$\boxed{0}$

$$x^4 + 3x^3 - 2x^2 + 3x + 1 = 0$$

$\div$ by x^2

$$x^2 + 3x - 2 + 3/x + 1/x^2 = 0$$

$$(x^2 + 1/x^2) + 3(x + 1/x) - 2 = 0$$

Put $y = x + 1/x$

$$x^2 + 1/x^2 = (x + 1/x)^2 - 2$$

$$= y^2 - 2$$

$$\cancel{1/x^2 \neq (x+1/x^2)}$$

$$(y^2 - 2) + 3y - 2 = 0$$

$$y^2 + 3y - 4 = 0$$

$$(y-1)(y+4) = 0$$

$$y = 1, -4$$

-4

$4 \quad -1$

3

$$x + 1/x = 1$$

$$\frac{x^2+1}{x} = 1$$

$$x^2 - x + 1 = 0$$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4\cdot1\cdot1}}{2\cdot1}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{1 \pm \sqrt{3}\, i}{2}$$

$$x + 1/x = -4$$

$$\frac{x^2+1}{x} = -4$$

$$x^2 + 1 = -4x$$

$$x^2 + 4x + 1 = 0$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4\cdot1\cdot1}}{2\cdot1}$$

$$= \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2}$$

$$x = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

$\therefore$ The roots are $-1, 1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$

$-2+\sqrt{3}, -2-\sqrt{3}$.

Note:

Let $x+1/x = y$

$$x^2+1/x^2 = (x+1/x)^2 - 2$$
$$= y^2-2$$
$$x^3+1/x^3 = (x+1/x)^3 - 3x^2 \cdot 1/x - 3x \cdot 1/x^2$$
$$= (x+1/x)^3 - 3x - 3/x$$
$$= (x+1/x)^3 - 3(x+1/x)$$
$$= y^3 - 3y.$$

(2) Solve $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$ —— (1)

Soln:

This is a reciprocal equation of odd degree with unlike signs.

$x=1$ is a root of (1)

1	1	-5	9	-9	5	-1	
	0	1	-4	5	-4	1	
	1	-4	5	-4	1	$\boxed{0}$	

$$x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$$

$(\div \text{ by } x^2)$

$$x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2} = 0$$
$$(x^2+1/x^2) - 4(x+1/x) + 5 = 0$$

Let $x+1/x = y$

$$x^2+1/x^2 = y^2-2$$
$$(y^2-2) - 4y + 5 = 0$$
$$y^2 - 4y + 3 = 0$$
$$(y-1)(y-3) = 0$$

$y = 1, \ y = 3$

$x + 1/x = 1$

$x^2 + 1 - x = 0$

$$x = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$x = \frac{1 \pm \sqrt{3}i}{2}$$

$x + 1/x = 3$

$x^2 + 1 - 3x = 0$

$$x = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

$$= \frac{3 \pm \sqrt{5}}{2}$$

$\therefore$ The roots are $1, \dfrac{1 \pm \sqrt{3}i}{2}, \dfrac{3 \pm \sqrt{5}}{2}$

3. Solve $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$ — (1)

Soln:

This is a reciprocal equation of even degree with unlike signs and middle term absent.

$x = \pm 1$ are the roots of (1).

1	6	-25	31	0	-31	25	-6	
	0	6	-19	12	12	-19	6	
-1	6	-19	12	12	-19	6	$\boxed{0}$	
	0	-6	25	-37	25	-6		
	6	-25	37	-25	6	$\boxed{0}$		

$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$

($\div$ by x^2)

$$6x^2 - 25x + 37 - \frac{25}{x} + \frac{6}{x^2} = 0$$

$6(x^2 + 1/x^2) - 25(x + 1/x) + 37 = 0$

Let $x + 1/x = y$

$x^2 + 1/x^2 = y^2 - 2$

$6(y^2 - 2) - 25y + 37 = 0$

$6y^2 - 25y + 25 = 0$

$(3y-5)(2y-5)=0$

$y = 5/3,\ 5/2$

150: $\frac{-10}{6}$, $\frac{-15}{6}$

$x + 1/x = \frac{5}{3}$

$\frac{x^2+1}{x} = \frac{5}{3}$

$3x^2 - 5x + 3 = 0$

$x = \frac{5 \pm \sqrt{25-4\cdot3\cdot3}}{2\cdot3}$

$= \frac{5 \pm \sqrt{25-36}}{6}$

$= \frac{5 \pm \sqrt{-11}}{6}$

$= \frac{5 \pm \sqrt{11}\, i}{6}$

$x + 1/x = 5/2$

$\frac{x^2+1}{x} = \frac{5}{2}$

4: $\frac{-4}{2}$, $\frac{-1}{2}$

$2x^2 - 5x + 2 = 0$

$(x-2)(2x-1)=0$

$x = 2,\ 1/2$.

$\therefore$ The roots are $1, -1, \frac{5 \pm \sqrt{11}\, i}{6}, 2, 1/2$

(4) Solve $2x^6 - 9x^5 + 10x^4 - 3x^3 + 10x^2 - 9x + 2 = 0$ —(1)

Soln:

This is a reciprocal equation of even degree with like sign and middle term.

Divide (1) by x^3,

$2x^3 - 9x^2 + 10x - 3 + \frac{10}{x} - \frac{9}{x^2} + \frac{2}{x^3} = 0$

$2(x^3 + 1/x^3) - 9(x^2 + 1/x^2) + 10(x + 1/x) - 3 = 0$

Put $x + 1/x = y$

$x^2 + 1/x^2 = y^2 - 2$

$x^3 + 1/x^3 = y^3 - 3y$

$2(y^3 - 3y) - 9(y^2 - 2) + 10y - 3 = 0$

$2y^3 - 6y - 9y^2 + 18 + 10y - 3 = 0$

$2y^3 - 9y^2 + 4y + 15 = 0$ — (2)

$f(y) = 2y^3 - 9y^2 + 4y + 15$

$f(1) = 2 - 9 + 4 + 15 \neq 0$

$f(-1) = -2 - 9 - 4 + 15 = 0$

$\therefore$ $y = -1$ is a root of (2)

$$\begin{array}{r|rrrr} -1 & 2 & -9 & 4 & 15 \\ & 0 & -2 & 11 & -15 \\ \hline & 2 & -11 & 15 & \boxed{0} \end{array}$$

$$\begin{array}{c} 30 \\ \wedge \\ \frac{-5}{2} \quad \frac{-6}{2} \end{array}$$

$2y^2 - 11y + 15 = 0$

$(y-3)(2y-5) = 0$

$\therefore$ $y = 3, 5/2$

$\therefore$ The roots of (2) are $-1, 3, 5/2$.

$x + 1/x = -1$

$x^2 + 1 + x = 0$

$x = \dfrac{-1 \pm \sqrt{1 - 4.1.1}}{2}$

$= \dfrac{-1 \pm \sqrt{-3}}{2}$

$= \dfrac{-1 \pm \sqrt{3}\, i}{2}$

$x + 1/x = 3$

$x^2 + 1 - 3x = 0$

$x = \dfrac{3 \pm \sqrt{9 - 4.1.1}}{2}$

$= \dfrac{3 \pm \sqrt{9-4}}{2}$

$= \dfrac{3 \pm \sqrt{5}}{2}$

$x + 1/x = 5/2$

$\dfrac{x^2+1}{x} = \dfrac{5}{2}$

$2x^2 + 2 - 5x = 0$

$$\begin{array}{c} 4 \\ \wedge \\ \frac{-4}{2} \quad \frac{-1}{2} \end{array}$$

$2x^2 - 5x + 2 = 0$

$(x-2)(2x-1) = 0$

$x = 2, 1/2$

$\therefore$ The roots of (1) are

$\dfrac{-1 \pm \sqrt{3}\, i}{2}$, $\dfrac{3 \pm \sqrt{5}}{2}$, $2, 1/2$

5 Solve $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$ — (1)

Soln:

This is a reciprocal equation of even degree with like signs and middle term.

Divide (1) by x^2,

$$4x^2 - 20x + 33 - \frac{20}{x} + \frac{4}{x^2} = 0$$

$$4\left(x^2 + 1/x^2\right) - 20\left(x + 1/x\right) + 33 = 0$$

Let $x + 1/x = y$

$$x^2 + 1/x^2 = y^2 - 2$$

$$4(y^2 - 2) - 20y + 33 = 0$$

100

$-\frac{10}{4}$ $\frac{-10}{4}$

$$4y^2 - 20y + 25 = 0$$

$$(2y - 5)(2y - 5) = 0$$

$$y = 5/2, 5/2$$

$$x + 1/x = 5/2$$

$$\frac{x^2+1}{x} = \frac{5}{2} \Rightarrow 2x^2 + 2 - 5x = 0$$

4

$\frac{-4}{2}$ $\frac{-1}{2}$

$$(x-2)(2x - 1/2) = 0$$

$$x = 2, 1/2$$

$\therefore$ The roots of (1) are

$2, 1/2, 2, 1/2$.

Unit-II

Descarte's Rule of Signs

Rule 1: The number of changes in signs of $f(x)=0$ is equal to the roots of positive real roots of $f(x)$.

Rule 2: The number of changes in signs of $f(-x)=0$ is equal to the roots of negative real roots of $f(x)$.

① Find the nature of the roots for the following equations

i) $x^3+2x+3=0$ —①

$$f(x) = x^3 + 2x + 3 = 0$$
$$+ \quad + \quad +$$

There is no change of signs in $f(x)=0$.

$\therefore$ $f(x)$ has no positive real roots.

$$f(-x) = (-x)^3 + 2(-x) + 3 = 0$$
$$= -x^3 - 2x + 3 = 0$$
$$- \quad - \quad +$$

There is one change of sign in $f(-x)=0$.

$\therefore$ $f(-x)$ has one negative real roots.

$f(x)=0$ is of degree 3. It has 3 roots.

$\therefore$ The remaining two roots are imaginary.

$\therefore$ The eqn. ① has one negative real roots and two imaginary roots.

ii) $x^5-6x^2-4x+5=0$ —①

$$f(x) = x^5 - 6x^2 - 4x + 5 = 0$$
$$+ \quad - \quad - \quad +$$

There are two changes of signs in $f(x)=0$.

$\therefore$ $f(x)$ has two positive real roots.

$f(-x) = (-x)^5 - 6(-x)^2 - 4(-x) + 5 = 0$

$= -x^5 - 6x^2 + 4x + 5 = 0$

$- \quad - \quad + \quad +$

$\therefore$ There is one change of sign in $f(x)=0$.

$\therefore$ $f(x)$ has one negative real root~~s~~.

$f(x)=0$ is of degree 5. It has 5 roots.

$\therefore$ The remaining two roots are imaginary.

$\therefore$ The eqn. ① has two positive real roots, one negative real root and two imaginary roots.

iii) $x^6 + 3x^2 - 5x + 1 = 0$ — ①

$f(x) = x^6 + 3x^2 - 5x + 1 = 0$

$+ \quad + \quad - \quad +$

There are two changes of sign in $f(x)=0$.

$\therefore$ $f(x)$ has two positive real roots.

$f(-x) = (-x)^6 + 3(-x)^2 \cancel{-} + 5(-x) + 1 = 0$

$= x^6 + 3x^2 + 5x + 1 = 0$

$+ \quad + \quad + \quad +$

There is no change of sign in $f(-x)=0$.

$\therefore$ $f(x)$ has no negative real roots.

$f(x)=0$ is of degree 6. It has 6 roots.

$\therefore$ The remaining four roots are imaginary.

$\therefore$ The eqn ① has two positive real roots, no negative real roots and four imaginary roots.

Newton's Method of Successive approximations
(or)
Newton's Raphson Method

The iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{where } n=0,1,2,3\ldots$$

1. Evaluate $\sqrt{12}$ upto two decimals using Newton's method.

$$\text{Let } x = \sqrt{12}$$

$$x^2 = 12$$

$$f(x) = x^2 - 12$$

$$f(3) = 9-12 = -3 = -ve$$

$$f(4) = 16-12 = 4 = +ve$$

A root lies between 3 and 4.

$$\text{Let } x_0 = \frac{3+4}{2} = 3.5$$

Newton's iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n=0,1,2,\ldots$$

$$f'(x) = 2x$$

First Iteration :

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 3.5 - \frac{f(3.5)}{f'(3.5)}$$

$$= 3.5 - \frac{(3.5)^2-12}{2(3.5)} = 3.46$$

Second Iteration :

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 3.46 - \frac{[(3.46)^2 - 12]}{2(3.46)}$$

$$x_2 = 3.46$$

$$x_1 = x_2 = 3.46$$

$\therefore$ The root is 3.46

(2) Find the real positive roots of $3x - \cos x - 1 = 0$ by Newton's method correct to 6 decimal places.

$$f(x) = 3x - \cos x - 1$$

$$f(0) = -2 = -ve$$

$$f(1) = 1.4596 = +ve$$

The root lies between 0 and 1.

Let $x_0 = \frac{0+1}{2} = 0.5$

The Newton's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f'(x) = 3 + \sin x$$

First Iteration : $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$$= 0.5 - \left[\frac{3(0.5) - \cos(0.5) - 1}{3 + \sin(0.5)}\right]$$

$$= 0.608519$$

Second Iteration : $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$= 0.608519 - \frac{f(0.608519)}{f'(0.608519)}$$

$$= 0.607102$$

Third Iteration:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.607102 - \frac{f(0.607102)}{f'(0.607102)}$$

$$= 0.607102$$

$$x_2 = x_3 = 0.607102$$

$\therefore$ The root is 0.607102.

③ Obtain Newton's iterative formula for finding $\sqrt{a}$ where 'a' is a positive number and hence find $\sqrt{5}$.

Soln: Let $x = \sqrt{a}$

$$x^2 = a \Rightarrow x^2 - a = 0$$

Let $f(x) = x^2 - a, \quad f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left[\frac{x_n^2 - a}{2x_n}\right] = x_n - \frac{x_n}{2} + \frac{a}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{a}{x_n}\right]$$

is the iterative formula to find $\sqrt{a}$.

To find $\sqrt{5}$:

Put $a = 5$

The root lies between 2 and 3.

Let $x_0 = \frac{2+3}{2} = 2.5$

$$x_1 = \frac{1}{2}\left[x_0 + \frac{5}{x_0}\right] = \frac{1}{2}\left[2.5 + \frac{5}{2.5}\right] = 2.25$$

$$x_2 = \frac{1}{2}\left[x_1 + \frac{5}{x_1}\right] = \frac{1}{2}\left[2.25 + \frac{5}{2.25}\right]$$

$$= 2.2361$$

$$x_3 = \frac{1}{2}\left[x_2 + \frac{5}{x_2}\right] = \frac{1}{2}\left[2.2361 + \frac{5}{2.2361}\right]$$

$$= 2.2361.$$

$$x_2 = x_3 = 2.2361.$$

$\therefore$ The root is 2.2361.

(4) Find the negative root of $f(x) = x^3 - 2x + 5 = 0$ Correct to five decimal places using Newton's method.

Soln: $f(x) = x^3 - 2x + 5$ — (1)

$$f(-x) = (-x)^3 - 2(-x) + 5$$

$$= -x^3 + 2x + 5 \quad \text{— (2)}$$

$f(0) = 5 = +ve$

$f(1) = +ve$

$f(2) = +ve$

$f(3) = -ve.$

The root of (2) lies between 2 and 3.

Newton's iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $x_0 = \frac{2+3}{2} = 2.5$

$f(x) = -x^3 + 2x + 5$

$f'(x) = -3x^2 + 2$

First Iteration:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{f(2.5)}{f'(2.5)}$$

$$= 2.5 - \frac{[-2.5^3 + 2(2.5) + 5]}{-3(2.5)^2 + 2}$$

$$x_1 = 2.16418$$

Second Iteration:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.16418 - \frac{f(2.16418)}{f'(2.16418)}$$

$$= 2.09714$$

Third Iteration:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.09714 - \frac{f(2.09714)}{f'(2.09714)}$$

$$= 2.09456$$

Fourth Iteration:

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.09456 - \frac{f(2.09456)}{f'(2.09456)}$$

$$= 2.09455$$

Fifth Iteration:

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 2.09455 - \frac{f(2.09455)}{f'(2.09455)}$$

$$= 2.09455.$$

$\therefore$ $x_4 = x_5 = 2.09455.$

$\therefore$ The approximate root of (2) is 2.09455

The approximate negative root of (1) is -2.09455

Horner's Method.

1. Calculate to two places of decimals the positive root of the equation $x^3 + 24x - 5 = 0$ using Horner's method.

Soln: $f(x) = x^3 + 24x - 5$ — (1)

$f(0) = -50 = -ve$

$f(1) = 1 + 24 - 50 = -ve$

$f(2) = 8 + 48 - 50 = +ve.$

A root lies between 1 and 2.

Diminish the root by 1,

1	1	0	24	−50
	0	1	1	25
	1	1	25	$\boxed{-25}$
	0	1	2	
	1	2	$\boxed{27}$	
	0	1		
	1	$\boxed{3}$		

The transformed equation is $x^3 + 3x^2 + 27x - 25 =$

Multiply the roots of the equation by 10,

$x^3 + 30x^2 + 2700x - 25000 = 0.$

Let $f_1(x) = x^3 + 30x^2 + 2700x - 25000$

$f_1(8) = -ve$

$f_1(9) = +ve.$

The root lies between 8 and 9.

Diminish the roots by 8,

8	1	30	2700	25000
	0	8	304	24032
	1	38	3004	$\boxed{-968}$
	0	8	368	
	1	46	$\boxed{3372}$	
	0	8		
	1	$\boxed{54}$		

The transformed equation is

$$x^3 + 54x^2 + 3372x - 968 = 0$$

Multiply the roots by 10,

$$x^3 + 540x^2 + 337200x - 968000 = 0$$

$$f_2(x) = x^3 + 540x^2 + 337200x - 968000$$

$$f_2(2) = -ve$$

$$f_3(3) = +ve$$

Diminish the roots by 2,

2	1	540	337200	-968000
	0	2	1084	676568
	1	542	338284	$\boxed{-291432}$
	0	2	1088	
	1	544	$\boxed{339372}$	
	0	2		
	1	$\boxed{546}$		

The transformed equation is

$$x^3 + 546x^2 + 339372x - 291432 = 0$$

Multiply the roots by 10,

$f_3(x) = x^3 + 5460x^2 + 33937600x - 291432000$

$f_3(8) = -ve$

$f_3(9) = +ve$

The root lies between 8 and 9.

The root correct to 2 decimal places is 1.83

(2) Find the real root of equation $x^3 + 6x - 2 = 0$ using Horner's method.

Soln:

$f(x) = x^3 + 6x - 2$

$f(0) = -ve$

$f(1) = +ve$

The real root lies between 0 and 1.

Multiply the roots by 10,

The transformed equation is

$x^3 + 600x - 2000 = 0$

$f_1(x) = x^3 + 600x - 2000$

$f_1(3) = -ve$

$f_1(4) = +ve$

The root lies between 3 and 4.

Diminish the roots by 3

3	1	0	600	-2000
	0	3	9	1827
	1	3	609	$\boxed{-173}$
	0	3	18	
	1	6	$\boxed{627}$	
	0	3		
	1	$\boxed{9}$		

The transformed equation is

$$x^3 + 9x^2 + 627x - 173 = 0$$

Multiply the roots by 10,

$$x^3 + 90x^2 + 62700x - 173000 = 0$$

$$f_2(x) = x^3 + 90x^2 + 62700x - 173000$$

$$f_2(2) = -ve$$

$$f_2(3) = +ve$$

The root lies between 2 and 3.

Diminish the roots by 2,

2	1	90	62700	-173000
	0	2	184	125768
	1	92	62884	$\boxed{-47232}$
	0	2	188	
	1	94	$\boxed{63072}$	
	0	2		
	1	$\boxed{96}$		

The transformed equation is

$x^3 + 96x^2 + 63072x - 47232 = 0$

Multiply the roots by 10,

$x^3 + 960x^2 + 6307200x - 47232000 = 0$

$f_3(x) = x^3 + 960x^2 + 6307200x - 47232000$

$f_3(7) = -ve$

$f_3(8) = +ve$

The root of this equation lies between 7 and 8.

The root correct to 2 decimal places is 0.38.

UNIT-III

Summation of Series

Binomial Series:

Binomial theorem for a Positive integral index

When n is a Positive integer

$$(x+a)^n = x^n + nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \dots + nC_r x^{n-r} a^r + \dots + a^n$$

There are $(n+1)$ terms in this expansion and the general term is given by

$$T_{r+1} = nC_r x^{n-r} a^r$$

Binomial Theorem for a rational index.

If n is a rational number and $-1<x<1$ then

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$+ \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots + \infty$$

This infinite Series is also called Binomial Series

$$(1+x)^{-p/q} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \dots + \infty$$

$$(1-x)^{-p/q} = 1 + \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3 + \dots + \infty$$

Results:

$$(1-x)^{-n} = 1+\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\dots\infty$$

$$(1+x)^{-n} = 1-\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+\dots\infty$$

$$(1+x)^{n} = 1+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots\infty$$

$$(1-x)^{n} = 1-\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+\dots\infty$$

Problems

1. Find the sum to infinity of the series

$$1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\dots\infty$$

Solution

Let $S = 1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\dots\infty$

$$= 1+\frac{2}{1!}\left(\frac{1}{6}\right)+\frac{2.5}{1.2}\left(\frac{1}{6}\right)^2+\frac{2.5.8}{1.2.3}\left(\frac{1}{6}\right)^3+\dots\infty \quad \to (1)$$

We know that

$$(1-x)^{-p/q} = 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\dots\infty \quad \to (2)$$

Comparing (1) & (2)

$p = 2$

$p+q=5 \Rightarrow q = 3$

$\frac{x}{q} = \frac{1}{6} \Rightarrow x = \frac{3}{6} = \frac{1}{2}$

$$\therefore S = (1-x)^{-p/q} = \left(1-\frac{1}{2}\right)^{-2/3} = \left(\frac{1}{2}\right)^{-2/3} = 2^{2/3}$$

$$= 4^{1/3}$$

2 Sum to infinity the Series $\frac{2.4}{3.6} + \frac{2.4.6}{3.6.9} + \frac{2.4.6.8}{3.6.9.12} + \dots \infty$

Soln

Let $S = \frac{2.4}{3.6} + \frac{2.4.6}{3.6.9} + \frac{2.4.6.8}{3.6.9.12} + \dots \infty$

$$= \frac{2.4}{1.2}\frac{1}{3^2} + \frac{2.4.6}{1.2.3}\left(\frac{1}{3^3}\right) + \frac{2.4.6.8}{1.2.3.4}\left(\frac{1}{3^4}\right) + \dots \infty$$

Add $1 + \frac{2}{1!}\left(\frac{1}{3}\right)$ on both sides

$$S + 1 + \frac{2}{1!}\left(\frac{1}{3}\right) = 1 + \frac{2}{1!}\left(\frac{1}{3}\right) + \frac{2.4}{2!}\left(\frac{1}{3}\right)^2 + \frac{2.4.6}{3!}\left(\frac{1}{3}\right)^3 + \dots \infty. \quad \text{(1)}$$

Comparing RHS of eqn (1) with $(1-x)^{-p/q}$ we get

$p = 2$

$p + q = 4$

$\Rightarrow q = 2.$

$\frac{x}{q} = \frac{1}{3}. \quad x = \frac{q}{3} \Rightarrow x = \frac{2}{3}$

$$S + 1 + \frac{2}{3} = \left(1 - \frac{2}{3}\right)^{-2/2}$$

$$S + \frac{5}{3} = \left(\frac{1}{3}\right)^{-1} = 3$$

$$S = 3 - \frac{5}{3}$$

$$S = \frac{9-5}{3}$$

$$\Rightarrow S = \frac{4}{3}$$

3. Sum to infinity the series $\frac{1.3}{2.4.6.8}+\frac{1.3.5}{2.4.6.8.10}+\cdots\infty$

$$S=\frac{1.3}{2.4.6.8}+\frac{1.3.5}{2.4.6.8.10}+\frac{1.3.5.7}{2.4.6.8.10\,12}+\cdots\infty$$

$$S(-1)(-3)=\frac{(-1)(-3)(1)(3)}{1.2.3.4}\left(\frac{1}{2^4}\right)+\frac{(-1)(-3).1.3.5}{1.2.3.4.5}\left(\frac{1}{2^5}\right)+\cdots\infty$$

$$3S=\frac{(-1)(-3)(1)(3)}{4!}\left(\frac{1}{2}\right)^4+\frac{(-1)(-3)(1)(3)(5)}{5!}\left(\frac{1}{2}\right)^5+\cdots\infty$$

Add the following term on both sides

$$1+\frac{(-3)}{1!}\left(\frac{1}{2}\right)+\frac{(-3)(-1)}{2!}\left(\frac{1}{2}\right)^2+\frac{(-3)(-1)(1)}{3!}\left(\frac{1}{2}\right)^3$$

$$3S+1+\frac{(-3)}{1!}\left(\frac{1}{2}\right)+\frac{(-3)(-1)}{2!}\left(\frac{1}{2}\right)^2+\frac{(-3)(-1)(1)}{3!}\left(\frac{1}{2}\right)^3=1+\frac{-3}{1!}\left(\frac{1}{2}\right)+\frac{(-3)(-1)}{2!}\left(\frac{1}{2}\right)^2$$
$$+\frac{(-3)(-1)(1)}{3!}\left(\frac{1}{2}\right)^3$$
$$+\frac{(-3)(-1)(1)(3)}{4!}\left(\frac{1}{2}\right)^4\cdots$$

Comparing with $(1-x)^{-p/q}$

$p=-3$ $\qquad$ $p+q=-1$

$\frac{x}{q}=\frac{1}{2}$ $\qquad$ $\Rightarrow q=2$

$\Rightarrow x=q/2$

$\Rightarrow x=1$

$$3S+1-\frac{3}{2}+\frac{3}{8}+\frac{3}{48}=(1-x)^{-p/q}=(1-1)^{+3/2}$$
$$=0$$

$$\Rightarrow \frac{3S+48-72+18+3}{48}=0$$

$$3s-\frac{3}{48}=0$$

$$\Rightarrow 3s=\frac{3}{48}$$

$$\boxed{S=\frac{1}{48}}$$

4. Find the Sum to Infinity the Series $2+\sum_{n=1}^{\infty}\frac{1}{3}\cdot\frac{1}{6^{n-1}}\frac{1.3.5\ldots(2n-1)}{n!}$

Soln

$$S_n=2+\frac{1}{3}\left[\frac{1}{1!}+\frac{1}{6}\frac{1.3}{2!}+\frac{1}{6^2}\frac{1.3.5}{3!}+\ldots\right]$$

$$=2+\frac{1}{3}\left[\frac{1}{1!}+\frac{1.3}{2!}\left(\frac{1}{6}\right)+\frac{1.3.5}{3!}\left(\frac{1}{6}\right)^2+\ldots\right]$$

$$S_n=2+\frac{6}{3}\left[\frac{1}{1!}\left(\frac{1}{6}\right)+\frac{1.3}{2!}\left(\frac{1}{6}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{6}\right)^3+\ldots\right]$$

$$=2\left[1+\frac{1}{1!}\left(\frac{1}{6}\right)+\frac{1.3}{2!}\left(\frac{1}{6}\right)^2+\frac{1.3.5}{3!}\left(\frac{1}{6}\right)^3+\ldots\right]\rightarrow (1).$$

$$(1-x)^{-p/q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+\ldots \rightarrow (2)$$

Comparing (1) & (2)

$$\frac{S_n}{2}=(1-x)^{-p/q}.$$

$p=1$

$p+q=3$

$q=2$

$\frac{x}{q}=\frac{1}{6}\Rightarrow x=\frac{1}{3}$

$$\frac{S_n}{2} = \left(1-\frac{1}{3}\right)^{-1/2}.$$

$$= \left(\frac{2}{3}\right)^{-1/2}$$

$$\frac{S_n}{2} = \left(\frac{3}{2}\right)^{1/2}$$

$$S_n = \frac{2\sqrt{3}}{\sqrt{2}}$$

$$S_n = \sqrt{2}\sqrt{3}$$

$$\Rightarrow S_n = \sqrt{6}$$

5 Show that $1 - \frac{n+x}{1!(1+x)} + \frac{(n+2x)(n-1)}{2!\,(1+x)^2} - \frac{(n+3x)(n-1)(n-2)}{3!\,(1+x)^3} + \cdots \infty = 0$

$$\Rightarrow \left[1 - \frac{n}{1!(1+x)} + \frac{n(n-1)}{2!\,(1+x)^2} - \frac{n(n-1)(n-2)}{3!\,(1+x)^3} + \cdots\right]$$

$$+ \left[\frac{-x}{1!(1+x)} + \frac{2x(n-1)}{2!\,(1+x)^2} - \frac{3x(n-1)(n-2)}{3!\,(1+x)^3} + \cdots\right]$$

$$\Rightarrow \left(1-\frac{1}{1+x}\right)^n - \frac{x}{1+x}\left[1 - \frac{n-1}{1!(1+x)} + \frac{(n-1)(n-2)}{2!\,(1+x)^2} + \cdots\right]$$

$$\Rightarrow \left(1-\frac{1}{1+x}\right)^n - \frac{x}{1+x}\left[1-\frac{1}{1+x}\right]^{n-1}$$

$$\left[(1-x)^n = 1 - \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\right]$$

$$\Rightarrow \left(\frac{1+x-1}{1+x}\right)^n - \frac{x}{1+x}\left(\frac{1+x-1}{1+x}\right)^{n-1}$$

$$\Rightarrow \left(\frac{x}{1+x}\right)^n - \left(\frac{x}{1+x}\right)^n = 0.$$

Hence the Proof

6 Assuming that the square and the higher power of x may be negleting and show that

$$\frac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} = 4 - \frac{10x}{3}$$

Soln

LHS

$$\frac{(1+x)^{1/2}(4-3x)^{3/2}}{(8+5x)^{1/3}} = \frac{(1+x)^{1/2}(4)^{3/2}\left(1-\frac{3x}{4}\right)^{3/2}}{8^{1/3}\left(1+\frac{5x}{8}\right)^{1/3}}$$

$$= \frac{(1+x)^{1/2}\, 8\left(1-\frac{3x}{4}\right)^{3/2}.}{2\left(1+\frac{5x}{8}\right)^{1/3}}$$

$$= 4\,(1+x)^{1/2}\left(1-\frac{3x}{4}\right)^{3/2}\left(1+\frac{5x}{8}\right)^{-1/3}$$

$$= 4\left(1+\tfrac{1}{2}x\right)\left(1-\tfrac{3}{2}\left(\tfrac{3x}{4}\right)\right)\left(1-\tfrac{1}{3}\left(\tfrac{5x}{8}\right)\right)$$

$$= 4\left[\left(1+\tfrac{x}{2}\right)\left(1-\tfrac{9x}{8}\right)\left(1-\tfrac{5x}{24}\right)\right]$$

$$= 4\left[\left(1-\tfrac{9x}{8}+\tfrac{x}{2}\right)\left(1-\tfrac{5x}{24}\right)\right]$$

negleting higher powers

$$= 4\left[1-\frac{9x}{8}+\frac{x}{2}-\frac{5x}{24}\right]$$

$$= 4\left[\frac{24-27x+12x-5x}{24}\right]$$

$$= 4\left[\frac{24-20x}{24}\right]$$

$$= 4\left[\left(1-\frac{20x}{24}\right]\right.$$

$$= 4 - 4\left(\frac{20x}{24}\right)$$

$$= 4 - \frac{10x}{3}$$

$$= RHS$$

Hence the Proof.

Exponential Series:

Definition:

For all values of x

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$$

The series $1 + \frac{x}{1!} + x^2/2! + x^3/3! + \dots \infty$ is called the exponential series

Results.

1 $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \infty$

2 $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^2}{4!} + \dots + \infty$

3 $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \infty$

4 $\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + 1/4! + \dots \infty$

5 $\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + 1/5! + \dots \infty$

Problems

1 Find the Coefficient of x^n in the expansion of e^{a+bx}

Soln

$$e^{a+bx} = e^a \cdot e^{bx}$$

$$= e^a\left[1 + \frac{bx}{1!} + \frac{(bx)^2}{2!} + \cdots + \frac{(bx)^n}{n!} + \cdots \infty\right]$$

The coefficient of x^n in $e^{a+bx} = e^a \cdot \frac{b^n}{n!}$

2 Prove that $\frac{e-1}{e+1} = \dfrac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots \infty}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots \infty}$

$$\text{RHS} = \frac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots \infty}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots \infty}$$

$$= \frac{\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots + \infty\right) - 1}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots \infty}$$

$$= \frac{\frac{e+e^{-1}}{2} - 1}{\frac{e-e^{-1}}{2}}$$

$$\frac{e^1+e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \cdots$$

$$\frac{e^1-e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

$$= \frac{\frac{e+e^{-1}-2}{\cancel{2}}}{\frac{e-e^{-1}}{\cancel{2}}}$$

$$= \frac{e + \frac{1}{e} - 2}{e - \frac{1}{e}}$$

$$= \frac{e^2 + 1 - 2e/\cancel{e}}{e^2 - 1/\cancel{e}}$$

$$= \frac{(e-1)^2}{(e+1)(e-1)} = \frac{e-1}{e+1} = LHS$$

Hence the Proof.

3 Prove that $\left(1 + \frac{1}{2!} + \frac{1}{4!} + \dots \infty\right)^2 - \left(1 + \frac{1}{3!} + \frac{1}{5!} + \dots \infty\right)^2 = 1$

Proof.

$$LHS = \left(1 + \frac{1}{2!} + \frac{1}{4!} + \dots \infty\right)^2 - \left(1 + \frac{1}{3!} + \frac{1}{5!} + \dots \infty\right)^2$$

$$= \left(\frac{e^1 + e^{-1}}{2}\right)^2 - \left(\frac{e - e^{-1}}{2}\right)^2$$

$$= \frac{(e^1 + e^{-1})^2}{4} - \frac{(e - e^{-1})^2}{4}$$

$$= \frac{e^2 + 2ee^{-1} + (e^{-1})^2 - (e^2 - 2ee^{-1} + (e^{-1})^2)}{4}$$

$$= \frac{\cancel{e^2} + 2 + \cancel{(e^{-1})^2} - \cancel{e^2} + 2 - \cancel{(e^{-1})^2}}{4} \qquad ee^{-1} = 1$$

$$= \frac{4}{4}$$

$$= 1$$

$= RHS$ Hence the Proof.

4. Sum to infinity the Series $1 + \frac{1+2}{2!} + \frac{1+2+2^2}{3!} + \cdots \infty$

Soln

The n^{th} term of the series is given by

$$T_n = \frac{1+2+2^2+\cdots+2^{n-1}}{n!}$$

$$T_n = \frac{2^{n-1}}{1 \cdot n!}$$

$$T_n = \frac{2^n}{n!} - \frac{1}{n!} \rightarrow (1)$$

Put $n = 1, 2, \ldots$

$$T_1 = \frac{2^1}{1!} - \frac{1}{1!}$$

$$T_2 = \frac{2^2}{2!} - \frac{1}{2!}$$

$$T_3 = \frac{2^3}{3!} - \frac{1}{3!}$$

$$\vdots$$

Adding all the $T_1, T_2 \ldots$

We get

$$S_\infty = \left(\frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)$$

$$S_\infty = \left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots - 1\right) - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots - 1\right)$$

$$= (e^2 - 1) - (e^1 - 1)$$

$$= e^2 - 1 - e + 1$$

$$= e^2 - e \Rightarrow S_\infty = e(e-1)$$

5 Sum to infinity the series $\frac{1^2}{1!}+\frac{1^2+2^2}{2!}+\frac{1^2+2^2+3^2}{3!}+\dots\infty$

The n^{th} term of the series is given by

$$T_n = \frac{n(n+1)(2n+1)}{6n!}$$

$$= \frac{n(n+1)(2n+1)}{6.n.(n-1)!}$$

$$T_n = \frac{(n+1)(2n+1)}{6(n-1)!} \rightarrow ①$$

Let $(n+1)(2n+1) = A + B(n-1) + C(n-1)(n-2) \rightarrow ②$

Put $n=1$ in ②

$(1+1)(2+1) = A$

$\Rightarrow \boxed{A=6}$

Put $n=2$ in ②

$(2+1)(4+1) = A+B$

$15 = 6+B$

$\Rightarrow \boxed{B=9}$

equating the coefficient of n^2 $\boxed{C=2}$

$$T_n = \frac{6+9(n-1)+2(n-1)(n-2)}{6(n-1)!}$$

$$= \frac{1}{(n-1)!} + \frac{3(n-1)}{(n-1)!} + \frac{(n-1)(n-2)}{3(n-1)!}$$

$$= \frac{1}{(n-1)!} + \frac{3\cancel{(n-1)}}{\cancel{(n-1)}(n-2)!} + \frac{\cancel{(n-1)}\cancel{(n-2)}}{3\cancel{(n-1)}\cancel{(n-2)}(n-3)!}$$

$$T_n = \frac{1}{(n-1)!} + \frac{3}{2(n-2)!} + \frac{1}{3(n-3)!}.$$

Put $n = 1, 2, 3 \dots$

$$T_1 = 1$$

$$T_2 = \frac{1}{1!} + \frac{3}{(2)}$$

$$T_3 = \frac{1}{2!} + \frac{3}{2(1!)} + \frac{1}{3}$$

$$T_4 = \frac{1}{3!} + \frac{3}{2.2!} + \frac{1}{3.1!}$$

$$\vdots$$

Adding $S_\infty = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots \infty\right) + \frac{3}{2}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots \infty\right)$

$$+ \frac{1}{3}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots \infty\right)$$

$$= e + \frac{3}{2}e + \frac{1}{3}e$$

$$= \frac{6e + 9e + 2e}{6}$$

$$S_\infty = \frac{17e}{16}$$

6 Find the sum to infinity the series $1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4} + \dots$

<u>Soln</u>

The n^{th} term of the series is given by

$$T_n = \frac{n^3}{n!} = \frac{n^3}{n(n-1)!} = \frac{n^2}{(n-1)!} \rightarrow (1)$$

Let $n^2 = A + B(n-1) + C(n-1)(n-2) \rightarrow (2)$

Put $n=1$

$A = 1$

Put $n=2$

$A+B=4$

$B=3$

Equating the coeff of n^2.

$C=1$

$$T_n = \frac{1+3(n-1)+1(n-1)(n-2)}{(n-1)!}$$

$$= \frac{1}{(n-1)!} + \frac{3(n-1)}{(n-1)!} + \frac{(n-1)(n-2)}{(n-1)!}$$

$$= \frac{1}{(n-1)!} + \frac{3(n-1)}{(n-1)(n-2)!} + \frac{(n-1)(n-2)}{(n-1)(n-2)(n-3)!}$$

$$= \frac{1}{(n-1)!} + \frac{3}{(n-2)!} + \frac{1}{(n-3)!}$$

Put $n=1,2,3 \ldots$

$T_1 = 1$

$T_2 = \frac{1}{1!} + 3$

$T_3 = \frac{1}{2!} + \frac{3}{1!} + 1$

$\vdots$

Adding all these we get

$$S_\infty = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\right) + 3\left(1 + \frac{1}{1!} + \ldots\right) + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$$

$S_\infty = e + 3e + e \quad = 5e$

7 $$\sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!}$$

Let $5n+1 = A+B(2n+1) \rightarrow (1)$

Put $2n+1=0$

$2n=-1$

$\Rightarrow n = -1/2$

Put $n=-1/2$ in eqn (1)

$$5(-1/2)+1 = A+B\left(2\times -1/2 +1\right)$$

$$-5/2 + 1 = A$$

$$\Rightarrow A = -3/2$$

Equating the coefficient of n

$$5 = 2B$$

$$\Rightarrow B = 5/2$$

$$T_n = \frac{-3/2 + 5/2\,(2n+1)}{(2n+1)!}$$

$$= \frac{-3}{2(2n+1)!} + \frac{5(2n+1)}{2(2n+1)!}$$

$$= \frac{-3}{2(2n+1)!} + \frac{5(2n+1)}{2(2n+1)\,2n!}$$

$$T_n = \frac{-3}{2(2n+1)!} + \frac{5}{2(2n)!}$$

$$S_\infty = \Sigma T_n$$

$$S_\infty = \sum_{n=0}^{\infty} \left[\frac{-3}{2(2n+1)!} + \frac{5}{2(2n)!} \right]$$

$$= \frac{-3}{2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} + \frac{5}{2} \sum_{n=0}^{\infty} \frac{1}{(2n)!}$$

$$= \frac{-3}{2}\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right) + \frac{5}{2}\left(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots\right)$$

$$= -\frac{3}{2}\left(\frac{e - e^{-1}}{2}\right) + \frac{5}{2}\left(\frac{e + e^{-1}}{2}\right)$$

$$= \frac{-3e}{4} + \frac{3e^{-1}}{4} + \frac{5e}{4} + \frac{5e^{-1}}{4}$$

$$= \frac{1}{4}(-3e + 3e^{-1} + 5e + 5e^{-1}$$

$$= \frac{1}{4}(2e + 8e^{-1})$$

$$= \frac{2}{4}(e + 4e^{-1})$$

$$= \frac{e}{2} + 2e^{-1}$$

$$S_\infty = \frac{e}{2} + \frac{2}{e}$$

Logarithmic Series

Formulaes

1 $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \infty$

2 $\log(1-x) = -\left(x + \frac{x^2}{2} + x^3/3 + \dots \infty\right)$

3 $-\log(1-x) = x + x^2/2 + x^3/3 + \dots \infty$

4 $\log\left(\frac{1+x}{1-x}\right) = 2\left(x + x^3/3 + x^5/5 + \dots \infty\right)$

5 $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \dots \infty$

6 $\frac{\log_e a}{\log_e b} = \log_b a.$

7 $\log(a/b) = -\log(b/a)$

Problems

1 Prove that $\log x = \frac{x-1}{x+1} + \frac{1}{2}\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\frac{x^3-1}{(x+1)^3} + \dots$

$$\frac{x-1}{x+1} = \frac{x}{x+1} - \frac{1}{x+1}$$

$$\frac{1}{2}\frac{x^2-1}{(x+1)^2} = \frac{1}{2}\frac{x^2}{(x+1)^2} - \frac{1}{2}\cdot\frac{1}{(x+1)^2}.$$

$$\frac{1}{3}\frac{x^3-1}{(x+1)^3} = \frac{1}{3}\frac{x^3}{(x+1)^3} - \frac{1}{3}\frac{1}{(x+1)^3}$$

$\vdots$

Adding

$$RHS = \left(\frac{x}{x+1} + \frac{1}{2}\frac{x^2}{(x+1)^2} + \frac{1}{3}\frac{x^3}{(x+1)^3} + \cdots\right)$$

$$-\left(\frac{1}{x+1} + \frac{1}{2}\frac{1}{(x+1)^2} + \frac{1}{3}\cdot\frac{1}{(x+1)^3} + \cdots\right)$$

$$= -\log\left(1 - \frac{x}{x+1}\right) - \left(-\log\left(1 - \frac{1}{x+1}\right)\right)$$

$$= -\log\left(\frac{x+1-x}{x+1}\right) + \log\left(\frac{x+1-1}{x+1}\right)$$

$$= -\log\left(\frac{1}{x+1}\right) + \log\left(\frac{x}{x+1}\right)$$

$$= -[\log 1 - \log(x+1)] + [\log x - \log(x+1)]$$

$$= \log(\cancel{x+1}) + \log x - \log(\cancel{x+1})$$

$$= \log x.$$

$$= LHS.$$

Hence the Proof.

2 Prove that $\frac{3}{10}\left[\log 10 + \frac{1}{2^7} + \frac{1}{2}\cdot\frac{3}{2^{14}} + \frac{1}{3}\cdot\frac{3^2}{2^{21}} + \cdots\right] = \log 2$

Proof

LHS

$$= \frac{3}{10}\left[\log 10 + \frac{1}{2^7} + \frac{1}{2}\cdot\frac{3}{2^{14}} + \frac{1}{3}\cdot\frac{3^2}{2^{21}} + \cdots\right]$$

$$= \frac{1}{10}\left[3\log 10 + \frac{3}{2^7} + \frac{1}{2}\cdot\frac{3^2}{2^{14}} + \frac{1}{3}\,\frac{3^3}{2^{21}} + \cdots\right]$$

$$= \frac{1}{10}\left[3\log 10 + \frac{3}{2^7} + \frac{1}{2}\left(\frac{3}{2^7}\right)^2 + \frac{1}{3}\left(\frac{3}{2^7}\right)^3 + \cdots\right]$$

$$= \frac{1}{10}\left[3\log 10 - \log\left(1-\frac{3}{2^7}\right)\right]$$

$$= \frac{1}{10}\left[3\log 10 - \log\left(\frac{128-3}{128}\right)\right]$$

$$= \frac{1}{10}\left[3\log 10 - \log\left(\frac{125}{128}\right)\right]$$

$$= \frac{1}{10}\left[\log 10^3 - \log\frac{125}{128}\right]$$

$$= \frac{1}{10}\left[\log 1000 - \log\frac{125}{128}\right]$$

$$= \frac{1}{10}\log\frac{1000}{125}\times 128$$

$$= \frac{1}{10}\log(8\times 128)$$

$$= \frac{1}{10}\log(2^3\times 2^7)$$

$$= \frac{1}{10}\log 2^{10}$$

$$= \frac{1}{10}\, 10\log 2$$

$$= \log 2$$

$$= RHS$$

Hence the Proof.

3 Show that $\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots = \frac{1}{2}\log\left(\frac{n+1}{n}\right)$

$$LHS = \frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \dots$$

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

$$= \frac{1}{2}\log\left(\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}\right)$$

$$= \frac{1}{2}\log\left(\frac{\frac{2n+1+1}{\cancel{2n+1}}}{\frac{2n+1-1}{\cancel{2n+1}}}\right)$$

$$= \frac{1}{2}\log\left(\frac{2n+2}{2n}\right)$$

$$= \frac{1}{2}\log\left(\frac{\cancel{2}(n+1)}{\cancel{2}n}\right)$$

$$= \frac{1}{2}\log\frac{n+1}{n}$$

$= RHS.$

Hence the Proof.

4 Sum the Series $\log_{10} e - \log_{10} 2e + \log_{10} 3e \dots$

$$\log_{10} e = \frac{1}{\log_e 10} = \frac{1}{\log 10}$$

$$\log_{10} 2e = \frac{1}{\log_e 10^2} = \frac{1}{\log 10^2} = \frac{1}{2\log 10}$$

$$\log_{10} 3e = \frac{1}{\log_e 10^3} = \frac{1}{\log 10^3} = \frac{1}{3\log 10}$$

$$\log_{10} e - \log_{10} 2e + \log_{10} 3e + \dots = \frac{1}{\log 10} - \frac{1}{2\log 10} + \frac{1}{3\log 10} - \dots$$

$$= \frac{1}{\log 10}\left(1 - \frac{1}{2} + \frac{1}{3} - \dots\right)$$

$$= \frac{\log 2}{\log 10}$$

5 Show that

$$1 + \left(\frac{1}{2} + \frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4} + \frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6} + \frac{1}{7}\right)\frac{1}{4^3} + \dots \infty = \log\sqrt{12}$$

Solution

$$LHS = \left(\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4^2} + \frac{1}{6}\cdot\frac{1}{4^3} + \dots \infty\right)$$

$$+ \left(1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\frac{1}{4^2} + \frac{1}{7}\frac{1}{4^3} + \dots \infty\right)$$

$$= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^3} + \dots \infty\right)$$

$$+ 2\left(\frac{1}{2} + \frac{1}{3}\frac{1}{2^3} + \frac{1}{5}\frac{1}{2^5} + \dots \infty\right)$$

$$= -\frac{1}{2}\log\left(1 - \frac{1}{4}\right) + \log\left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right)$$

$$= -\frac{1}{2}\log\left(\frac{3}{4}\right) + \log\left(\frac{\frac{3}{\cancel{2}}}{\frac{1}{\cancel{2}}}\right)$$

$$= \frac{1}{2}\log\left(\frac{4}{3}\right) + \log 3$$

$$= \log\left(\frac{4}{3}\right)^{1/2} + \log 3$$

$$= \log\sqrt{\frac{4}{3}\cdot 3} = \log\sqrt{12}$$

6 Show that $\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \dots \infty = \log 2 - \frac{1}{2}$

Solution :

The n^{th} term of the series is

$$T_n = \frac{1}{(2n-1)(2n)(2n+1)}$$

Let $\frac{1}{(2n-1)(2n)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n} + \frac{C}{2n+1}$

$$1 = A(2n)(2n+1) + B(2n-1)(2n+1) + C(2n)(2n-1)$$

Put $n = \frac{1}{2}$

$$1 = A \cdot 1 \cdot 2 \Rightarrow A = \frac{1}{2}.$$

Put $n = 0$

$$1 = -B \Rightarrow B = -1$$

Put $n = -\frac{1}{2}$

$$1 = C(-2)(-1) = 2C \Rightarrow C = \frac{1}{2}.$$

$$T_n = \frac{\frac{1}{2}}{2n-1} - \frac{1}{2n} + \frac{\frac{1}{2}}{2n+1}$$

Put $n = 1, 2, 3 \dots$

$$T_1 = \frac{\frac{1}{2}}{1} - \frac{1}{2} + \frac{\frac{1}{2}}{3}$$

$$T_2 = \frac{\frac{1}{2}}{3} - \frac{1}{4} + \frac{\frac{1}{2}}{5}$$

$$T_3 = \frac{\frac{1}{2}}{5} - \frac{1}{6} + \frac{\frac{1}{2}}{7}$$

$\vdots$

$$T_n = \frac{-2}{2n} + \frac{1}{2n-1} + \frac{1}{2n+1}$$

Put $n = 1, 2, 3 \ldots$

$$T_1 = \frac{-2}{2} + \frac{1}{1} + \frac{1}{3} = \frac{1}{1} - 1 + \frac{1}{3}$$

$$T_2 = -\frac{2}{4} + \frac{1}{3} + \frac{1}{5} = \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$$

$$T_3 = -\frac{2}{6} + \frac{1}{5} + \frac{1}{7} = \frac{1}{5} - \frac{2}{6} + \frac{1}{7}.$$

$\vdots$

Adding all these we get

$$S_\infty = 1 + \left(-1 + \left(\frac{1}{3} + \frac{1}{3}\right) - \frac{2}{4} + \left(\frac{1}{5} + \frac{1}{5}\right) - \frac{2}{6} \cdots\right)$$

$$= 1 + \left(-1 + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} \cdots\right)$$

$$= 1 + 2\left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right)$$

$$= 1 + 2\left(1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots - 1\right)$$

$$= 1 + 2\log 2 - 2$$

$$= 2\log 2 - 1$$

Hence the Proof.

Adding $S_\infty = \frac{1}{2} + \left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right)$

$$= \frac{1}{2} + \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots -1\right]$$

$$= \frac{1}{2} + (\log 2 - 1)$$

$$= \log 2 - \frac{1}{2}$$

7. $\frac{1}{1\cdot1\cdot3} + \frac{1}{2\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \cdots \infty = 2\log 2 - 1$

$$T_n = \frac{1}{n(2n-1)(2n+1)}$$

$$T_n = \frac{2}{2n(2n-1)(2n+1)}$$

$$\frac{2}{2n(2n-1)(2n+1)} = \frac{A}{2n} + \frac{B}{2n-1} + \frac{C}{2n+1}$$

$$2 = A(2n-1)(2n+1) + B(2n)(2n+1) + C(2n)(2n-1)$$

Put $n=0$

$$2 = A(-1)(1) \Rightarrow A = -2.$$

Put $n = \frac{1}{2}$

$$2 = B\left(2\times\frac{1}{2}\right)\left(2\times\frac{1}{2}+1\right)$$

$$2 = B(1)(2)$$

$$\Rightarrow B = 1$$

Put $n = -\frac{1}{2}$.

$$2 = C\left(2\times-\frac{1}{2}\right)\left(2\times-\frac{1}{2}-1\right)$$

$$\Rightarrow C = 1$$

UNIT - IV
MATRICES

Definition:

Singular and Non Singular Matrix

A Square Matrix A is said to be singular if $|A| = 0$

A Square Matrix A is said to be non Singular if $|A| \neq 0$

Symmetric Matrix

A Square Matrix $A = [a_{ij}]$ is said to be Symmetric if $[a_{ij}] = [a_{ji}]$ (or) $A = A^T$

Eg
$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 0 & 3 \\ -5 & 3 & -1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 0 & 3 \\ -5 & 3 & -1 \end{bmatrix}$$

Skew Symmetric Matrix

A Square Matrix $A = [a_{ij}]$ is Said to be Skew symmetric if $[a_{ij}] = -[a_{ji}]$ or $A = -A^T$

Eg

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

$$-A^T = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

1. If A and B are both Symmetric, then AB is Symmetric if and only if A and B are Commutative.

Proof.

Given A and B are Symmetric

$A = A'$

$B = B'$

To Prove AB is symmetric

$(AB)' = B'A'$

$= BA$

$(AB)' = AB$ $(\because$ A and B are Commutative $AB = BA)$

$\therefore$ AB is Symmetric

Also if AB is symmetric

$(AB)' = AB \rightarrow (1)$

$(AB)' = B'A'$

$(AB)' = BA \rightarrow (2)$ $(\because A = A', B = B')$

from (1) & (2)

$AB = BA$

$\therefore$ AB is Commutative

2 If A and B are symmetric show that A+B is Symmetric

Proof: A and B are Symmetric

$A = A', \quad B = B'$

To Prove A+B is Symmetric

$\because A = A' \quad B = B'$

$(A+B)' = A' + B' = A+B$ $\therefore$ A+B is Symmetric

3 Show that every square matrix can be uniquely expressed as the sum of a symmetric and a skew symmetric matrix.

Proof

Let A be a square matrix

$$A = \tfrac{1}{2}(A+A') + \tfrac{1}{2}(A-A') \longrightarrow (1)$$

Now

$$(A+A')' = A' + (A')'$$

$$= A' + A$$

$$= A + A'$$

$\therefore$ $A+A'$ is symmetric

$$(A-A')' = A' - (A')'$$

$$= A' - A$$

$$= -(A-A')$$

$\therefore$ $A-A'$ is skew symmetric

Let $A = P + Q$

Where $P = \tfrac{1}{2}(A+A')$

$Q = \tfrac{1}{2}(A-A')$

$\therefore$ Any square matrix can be expressed as the sum of the symmetric and a skew symmetric matrix.

To show that the representation is unique

$$A = R + S \longrightarrow (2)$$

Where R is symmetric $\Rightarrow R = R'$

S is skew symmetric $\Rightarrow -S = S'$

$$A' = (R+S)'$$
$$= R' + S'$$
$$A' = R - S \quad \rightarrow (3)$$

(2) + (3)

$$A + A' = (R+S) + (R-S)$$
$$A + A' = 2R$$
$$\frac{1}{2}(A+A') = R = P$$

(2) − (3)

$$A - A' = (R+S) - (R-S)$$
$$= R + S - R + S$$
$$= 2S$$
$$\frac{1}{2}(A-A') = S = Q$$

∴ There is only one way of expressing a square matrix as the sum of a symmetric and skew symmetric matrices.

Conjugate of a Matrix

The matrix obtained from any given matrix A on replacing its elements by corresponding complex number is called the conjugate of A and is denoted by $\bar{A}$.

Eg:
$$A = \begin{bmatrix} 3+2i & 4-i \\ 1+i & 2+3i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 3-2i & 4+i \\ 1-i & 2-3i \end{bmatrix}$$

Hermition Matrix:

A Square matrix $A=[a_{ij}]$ is Said to be hermitian if $[a_{ij}]=[\overline{a_{ji}}]^T$. it is denoted by $A=A^*$

Eg

$$A=\begin{bmatrix}3 & 4+5i\\4-5i & 6\end{bmatrix} \quad \bar{A}=\begin{bmatrix}3 & 4-5i\\4+5i & 6\end{bmatrix} \quad \bar{A}^T=\begin{bmatrix}3 & 4+5i\\4-5i & 6\end{bmatrix}$$

$\therefore\ A=\bar{A}^T$ ie $A=A^*$

Skew Hermitian Matrix:

A Square matrix $A=[a_{ij}]$ is said to be Skew hermitian if $[a_{ij}]=-[\overline{a_{ji}}]$ for All i and j it is denoted by $A=-A^*$

Eg

$$\begin{bmatrix}0 & 3-4i\\-3-4i & 0\end{bmatrix}$$

1 Theorem

If A and B are hermitian show that $AB+BA$ is hermitian $AB-BA$ is Skew hermitian.

Given

A and B are Hermitian

$\therefore\ A=A^*,\ B=B^*$

To Prove $AB+BA$ is hermitian

$$\begin{aligned}(AB+BA)^* &= (AB)^*+(BA)^*\\ &= B^*A^*+A^*B^*\\ &= BA+AB\\ &= AB+BA\end{aligned}$$

$\therefore$ $AB+BA$ is Hermitian

To Prove AB-BA is skew hermitian

$$(AB-BA)^* = (AB)^* - (BA)^*$$
$$= B^*A^* - A^*B^*$$
$$= BA - AB$$
$$= -(AB-BA)$$

$\therefore$ AB-BA is Skew Hermitian.

2. <u>Theorem</u>

Show that B^*AB is skew hermitian or hermitian according as A is skew hermitian or hermitian.

Proof

A is hermitian $A = A^*$

$$(B^*AB)^* = B^*A^*(B^*)^*$$
$$= B^*A^*B$$
$$= B^*AB \qquad \because A = A^*$$

$$\therefore (B^*AB)^* = B^*AB$$

$\therefore$ B^*AB is hermitian

A is skew hermitian $A^* = -A$

$$(B^*AB)^* = B^*A^*(B^*)^*$$
$$= B^*A^*B$$
$$= B^*(-A)B \qquad \because A^* = -A$$
$$= -(B^*AB)$$

$\therefore$ B^*AB is Skew Hermitian.

Orthogonal and Unitary Matrices

Orthogonal Matrix:

A square matrix A is said to be orthogonal if $AA' = A'A = I$

Unitary Matrix

A Square matrix A is said to be unitary if $A^*A = AA^* = I$

Problems

1 Prove that the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

TO Prove A is orthogonal $AA' = I$

$$A' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal.

2 Show that $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary.

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \qquad A' = \begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{-1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$\overline{A'} = A^* = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-(1+i)}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$A^*A = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ \frac{-(1+i)}{2} & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$\therefore$ A is unitary.

Rank of The Matrix:

The number 'r' is said to be the rank of the matrix A if A possesses atleast minor of the order 'r' which does not vanish.

Every minor of A of order r+1 and higher order vanish then $\rho(A) = r$.

1 Find the rank of the matrix $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \\ -3 & 1 & -2 \end{bmatrix}$

Soln

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & -4 \\ -3 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

Since all the minors of order 3 and 2 vanish

$\rho(A) \neq 3$, $\rho(A) \neq 2$

$\therefore$ $\rho(A) = 1$

2 Find the rank of the matrix $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

Soln

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 6 & 1 & 3 & 8 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 = R_4 - R_3 \end{array}$$

$$= \begin{array}{c} \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$= \begin{array}{c} \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \to R_3 - R_2 \\ R_4 \end{array}$$

all the minor of order 4 and 3 vanished

2×2 order are not vanished

$\therefore \ \rho(A) = 2$.

Test for consistency:

Suppose we are given m equations and n unknowns

Step 1: write down the coefficient matrix A

Step 2: Write down the augmented matrix [A, B]

Step 3: Apply elementary transformation to find the ranks of A and [A, B]

Case (i):

If rank of A < rank [A, B]

the equations are inconsistent and they have no solutions

case (ii):

(i) If rank of A = rank [A, B]

the equations are consistent

(ii) If rank of A = rank [A, B] = r = n

the solution is unique

(iii) If rank of A = rank [A, B] = r < n

Infinite number of solutions.

1. Show the System of equation $x-3y-8z=-10$; $3x+y-4z=0$, $2x+5y+6z=13$ are Consistent and Solve them.

Soln

$$A=\begin{bmatrix}1 & -3 & -8\\ 3 & 1 & -4\\ 2 & 5 & 6\end{bmatrix} \qquad B=\begin{bmatrix}-10\\ 0\\ 13\end{bmatrix}$$

$$[A, B] = \begin{bmatrix}1 & -3 & -8 & -10\\ 3 & 1 & -4 & 0\\ 2 & 5 & 6 & 13\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & -3 & -8 & -10\\ 0 & 10 & 20 & 30\\ 0 & 11 & 22 & 33\end{bmatrix}\begin{array}{l}R_1\\ R_2 \to R_2-3R_1\\ R_3 \to R_3-2R_1\end{array}$$

$$\sim \begin{bmatrix}1 & -3 & -8 & -10\\ 0 & 1 & 2 & 3\\ 0 & 1 & 2 & 3\end{bmatrix}\begin{array}{l}R_1\\ R_2/10\\ R_3/11\end{array}$$

$$\sim \begin{bmatrix}1 & -3 & -8 & -10\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{array}{l}R_1\\ R_2\\ R_3-R_2.\end{array}$$

$\rho(A)=2=\rho(A, B)=2<3$

$\therefore$ The equation is consistent and has infinite number of solution

$$x-3y-8z=-10$$

$$y+2z=3$$

Take $z=k$.

$$y=3-2k.$$

$x = -10 + 3y + 8z$

$= -10 + 3(3-2k) + 8k.$

$= -10 + 9 - 6k + 8k$

$= -1 + 2k.$

$x = 2k-1$

$y = 3-2k$

$z = k$

When $k = 1$ $\quad x = 1$

$y = 1$

$z = 1$

When $k = 2$

$x = 3$

$y = -1$

$z = 2$ where k is any arbitraty value

$\therefore$ This system has infinite number of solutions

Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

$|A - \lambda I| = 0$ where λ is a scalar

Problem:

1 Determine the characteristic roots of the matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda(\lambda^2-1) - 1[-\lambda+2] + 2[-1+2\lambda] = 0$

$\Rightarrow -\lambda^3 + \lambda + \lambda - 2 - 2 + 4\lambda = 0$

$\Rightarrow -\lambda^3 + 6\lambda - 4 = 0$

$\times$ by $(-)$ $\lambda^3 - 6\lambda + 4 = 0$

$\lambda = 2$ is a root

$$\begin{array}{c|cccc} 2 & 1 & 0 & -6 & 4 \\ & 0 & 2 & 4 & -4 \\ \hline & 1 & 2 & -2 & \boxed{0} \end{array}$$

$\lambda^2 + 2\lambda - 2 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{12}}{2}$$

$$\lambda = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}$$

$\therefore$ The characteristic roots are $2, -1 \pm \sqrt{3}$

2 Find the characteristic roots of the orthogonal matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that they are of unit modulus.

$$|A-\lambda I|=0$$

$$\left|\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}-\begin{pmatrix}\lambda & 0 \\ 0 & \lambda\end{pmatrix}\right|=0$$

$$\begin{vmatrix}\cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda\end{vmatrix}=0$$

$$(\cos\theta-\lambda)^2+\sin^2\theta=0$$

$$\cos^2\theta-2\lambda\cos\theta+\lambda^2+\sin^2\theta=0$$

$$\lambda^2-2\lambda\cos\theta+1=0$$

This is the Characteristic equation of the given orthogonal matrix

$$a=1,\quad b=-2\cos\theta,\quad c=1$$

$$\lambda=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$

$$=\frac{2\cos\theta\pm2\sqrt{-(1-\cos^2\theta)}}{2}$$

$$\lambda=\frac{2\cos\theta\pm2i\sin\theta}{2}$$

$$\lambda=\cos\theta\pm i\sin\theta$$

$$|\lambda|=\sqrt{\cos^2\theta+\sin^2\theta}$$

$$=\sqrt{1}$$

$$|\lambda|=1$$

3 Find the characteristic equation of matrix $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ and verify that it satisfied by A also find A^{-1}

Soln

$$A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$|A-\lambda I|=0$

$$\left|\begin{pmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}\right|=0$$

$$\begin{vmatrix}2-\lambda & -1 & 1\\ -1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}=0$$

$\Rightarrow 2-\lambda\left[(2-\lambda)^2-1\right]+1\left[-(2-\lambda)+1\right]+1\left[1-(2-\lambda)\right]=0$

$\Rightarrow 2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda+\lambda-1+\lambda-1=0$

$\Rightarrow -\lambda^3+6\lambda^2-9\lambda+4=0$

$\times$ by$(-)$

$$\lambda^3-6\lambda^2+9\lambda-4=0$$

$A^3-6A^2+9A-4I=0$

$A^3=A^2\cdot A$

$A^2=A\cdot A\cdot$

$$A^2=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$A^2=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$A^3 = A^2 \cdot A$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$A^3 - 6A^2 + 9A - 4I = 0$

$$\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 22-36+18-4 & -21-30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 - 6A^2 + 9A - 4I = 0 \rightarrow ①$

$\times$ by A^{-1}

$A^2 - 6A + 9AA^{-1} - 4IA^{-1} = 0$

$A^2 - 6A + 9I - 4A^{-1} = 0$

$A^{-1} = \frac{1}{4}[A^2 - 6A + 9I]$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & 1 & 3 \end{bmatrix}$$

Eigen values and Eigen vectors:

1 Find the Eigen values and Eigen vectors of $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{bmatrix}$

Soln

The characteristic equation is $|A-\lambda I|=0$

$$\left| \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 1-\lambda & 1 \\ -7 & 2 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(1-\lambda)(-3-\lambda)-2]-2[2(-3-\lambda)+7]+0=0$$

$$\Rightarrow (2-\lambda)[\lambda^2+2\lambda-5]-2[-2\lambda+1]=0$$

$$2\lambda^2+4\lambda-10-\lambda^3-2\lambda^2+5\lambda+4\lambda-2=0$$

$$-\lambda^3+13\lambda-12=0$$

$$\Rightarrow \lambda^3+0\lambda^2-13\lambda+12=0$$

$\lambda=1$ satisfies the above equation

$\therefore \lambda=1$ is a root

$$\begin{array}{c|cccc} 1 & 1 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & \boxed{0} \end{array}$$

$$\lambda^2+\lambda-12=0$$

$$(\lambda+4)(\lambda-3)=0$$

$$\Rightarrow \lambda=-4, \quad \lambda=3$$

$\therefore$ The Eigen values are $-4, 1, 3$

To find the Eigen vector

$$[A - \lambda I][x] = 0$$

Case i) When $\lambda = -4$

$$\left[\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 6 & 2 & 0 \\ 2 & 5 & 1 \\ -7 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$6x_1 + 2x_2 + 0x_3 = 0$

$2x_1 + 5x_2 + x_3 = 0$

$-7x_1 + 2x_2 + x_3 = 0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 2 & 0 & 6 & 2 \\ 5 & 1 & 2 & 5 \end{array}$$

$$\frac{x_1}{2-0} = \frac{x_2}{0-6} = \frac{x_3}{30-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-6} = \frac{x_3}{26} \Rightarrow \frac{x_1}{1} = \frac{x_2}{-3} = \frac{x_3}{13}$$

$\therefore$ The Eigen vetor is $X_1 = \begin{pmatrix} 1 \\ -3 \\ 13 \end{pmatrix}$

Case (ii) When $\lambda = 1$

$$\left[\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 + 0x_3 = 0$

$2x_1 + 0x_2 + x_3 = 0.$

$-7x_1 - 2x_2 - 4x_3 = 0$

$$\frac{x_1}{2-0} = \frac{x_2}{0-1} = \frac{x_3}{0-4}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-4}$$

$\therefore$ The Eigen vector $X_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$

Case (iii) When $\lambda = 3$

$$\left[\begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ -7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + 2x_2 + 0x_3 = 0$$
$$2x_1 - 2x_2 + x_3 = 0$$
$$-7x_1 + 2x_2 - 6x_3 = 0$$

$$\begin{array}{cccc} x_1 & & x_2 & & x_3 \\ 2 & 0 & -1 & 2 \\ -2 & 1 & 2 & -2 \end{array}$$

$$\frac{x_1}{2-0} = \frac{x_2}{0+1} = \frac{x_3}{2+4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$\therefore$ The Eigen vector $X_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\therefore$ The Eigen values are $-4, 1, 3$

The Eigen vectors are $\begin{bmatrix} 1 \\ -3 \\ 13 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

2 Find all characteristic vectors of $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = 0$

Soln

The characteristic equation is $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 - 4 = 0$$

$$9 - 6\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 5)(\lambda - 1) = 0$$

$$\lambda = 5, 1$$

$\therefore$ The characteristic values are 1, 5

To find characteristic vector (or) Eigen vector

$$[A - \lambda I] X = 0$$

Case i) when $\lambda = 1$

$$\left[\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2x_1 + 2x_2 = 0$$

$$2x_1 + 2x_2 = 0$$

$x_1 = k$, $x_2 = -k$ Satisfies all values of k.

$\therefore$ the eigen vector $k\begin{pmatrix}1\\-1\end{pmatrix} = X_1$

Case (ii) when $\lambda = 5$

$$\left[\begin{pmatrix}3 & 2\\2 & 3\end{pmatrix} - \begin{pmatrix}5 & 0\\0 & 5\end{pmatrix}\right]\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}-2 & 2\\2 & -2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$-2x_1 + 2x_2 = 0$$
$$2x_1 - 2x_2 = 0.$$

$x_1 = k$, $x_2 = k$ Satisfies the equations for all values of k.

$\therefore$ The Eigen vector $X_2 = k\begin{bmatrix}1\\1\end{bmatrix}$

The Ch. value $= 1, 5$

The Ch. vectors $= X_1 = k\begin{bmatrix}1\\-1\end{bmatrix}$ $X_2 = k\begin{bmatrix}1\\1\end{bmatrix}$

Similarity of Matrices:

Let A and B be two Square matrices of Same order then the matrix B is said to be Similar to the matrix A. If there exist a non-Singular matrix P ie $B = P^{-1}AP$.

Diagonalization

A matrix 'A' is said to be diagonalizable if it is similar to a diagonal matrix then there exist a non-singular matrix P i.e $D = P^{-1}AP$

Problem

1. Diagonalize the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(5-\lambda)(1-\lambda)-1] - 1[1(1-\lambda)-3] + 3[1-3(5-\lambda)] = 0$

$\Rightarrow (1-\lambda)[5-5\lambda-\lambda+\lambda^2-1] + [-1+\lambda+3] + 3[1-15+3\lambda] = 0$

$\Rightarrow \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 + 3 - 45 + 9\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 7\lambda^2 - 36 = 0$

$\therefore$ The Ch. eqn is $\lambda^3 - 7\lambda^2 + 36 = 0$

To find Eigen values

$\lambda = 3$ is a root

$$\begin{array}{r|rrrr} 3 & 1 & -7 & 0 & 36 \\ & 0 & 3 & -12 & -36 \\ \hline & 1 & -4 & -12 & \boxed{0} \end{array}$$

$$\lambda^2 - 4\lambda - 12 = 0$$

$$(\lambda - 6)(\lambda + 2) = 0$$

$$\lambda = 6, \quad \lambda = -2.$$

$\therefore$ The Eigen values are $-2, 3, 6$

To Find Eigen vectors:

$$[A - \lambda I]X = 0$$

Case (i) When $\lambda = -2$

$$\left[\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 + 3x_3 = 0$$

$$x_1 + 7x_2 + x_3 = 0$$

$$3x_1 + x_2 + 3x_3 = 0$$

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

When $\lambda = 3$

$$\left[\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 + 3x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$3x_1 + x_2 - 2x_3 = 0$$

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & -2 & 1 \\ 2 & 1 & 1 & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$\therefore$ The eigen vectors $X_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

Case (iii)

When $\lambda = 6$

$$\left[\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-5x_1 + x_2 + 3x_3 = 0$$
$$x_1 - x_2 + x_3 = 0$$
$$3x_1 + x_2 - 5x_3 = 0$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ 1 & 3 & -5 & 1 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

To find
$D = P^{-1} A P$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P$$

$$|P| = -1[1+2] + 1[0-2] + 1[0-1]$$
$$= -3 - 2 - 1$$
$$= -6$$

$$\operatorname{adj} P = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 2 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} (1+2) & (2-0) & (0-1) \\ (-1+1) & (-1-1) & (-1-1) \\ (-2-1) & (0+2) & (-1+0) \end{bmatrix}^T$$

$$= \begin{pmatrix} 3 & 2 & -1 \\ 0 & -2 & -2 \\ -3 & 2 & -1 \end{pmatrix}^T$$

$$\text{adj } P = \begin{pmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{|P|} \text{adj } P$$

$$= \frac{1}{-6} \begin{pmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ -1 & -2 & -1 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$D = P^{-1} A P$$

$$= \frac{1}{6} \begin{pmatrix} -3 & 0 & 3 \\ -2 & 2 & -2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -3+0+9 & -3+0+3 & -9+0+3 \\ -2+2-6 & -2+10-2 & -6+2-2 \\ 1+2+3 & 1+10+1 & 3+2+1 \end{bmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 0 & -6 \\ -6 & 6 & -6 \\ 6 & 12 & 6 \end{pmatrix}\begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -6+0-6 & -6+0+6 & 6+0-6 \\ 6+0-6 & 6+6+6 & -6+12-6 \\ -6+0+6 & -6+12-6 & 6+24+6 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -12 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{pmatrix}$$

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

UNIT V

Elementary Numbers Theory

Prime Numbers:

A number which is divisible by 1 and the number itself is called a Prime number

Eg: 2, 3, 5, 7, 11, ...

Composite Number:

A number which is not prime or a number which has the divisor except one and itself is called a composite number.

Results

1. If a divides b and b divides c then a divides c

$$a/b \ \& \ b/c \Rightarrow a/c$$

2. If 'd' divides a and b then d divides $ma \pm nb$ where $m, n \in N$.

$$d \text{ divides } a \Rightarrow a = kd$$

$$d \text{ divides } b \Rightarrow b = ld$$

$$ma \pm nb = mkd + nld$$

$$= (mk \pm nl)d$$

$\therefore d$ divides $ma \pm nb$

Greatest Common divisor:

If a and b are two numbers then the greatest of all the numbers which divide both a and b is called the greatest common divisor of the two numbers.

If the GCD of two numbers is unity then the numbers are said to be prime to each other or relatively prime.

Result:

Every Composite number can be resolved into Prime factor and this can be done only in one way

Eg

$$\begin{array}{r|l} 2 & 72 \\ 2 & 36 \\ 2 & 18 \\ 3 & 9 \\ 3 & 3 \\ & 1 \end{array}$$

$72 = 2^3 \times 3^2$.

Divisors of a given number be

$$N = p^a . q^b . r^c$$

where p, q, r are Primes

a, b, c are integers.

Sum of divisors: $\dfrac{p^{a+1}-1}{p-1} \cdot \dfrac{q^{b+1}-1}{q-1} \cdot \dfrac{r^{c+1}-1}{r-1}$

Number of divisors of $N = (a+1)(b+1)(c+1)$

Product of divisors of $N = N^{1/2 (a+1)(b+1)(c+1)}$

Perfect Number:

A number is called a Perfect number if the sum of its divisors excluding the number is equal to the number.

Euler's function.

The number of integers less than N and Prime to N is called Euler's functions and is denoted by $\phi(N)$

$$\phi(N) = N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)$$

Sum of all poisitive integer less than N and prime to N is given by $\frac{N\phi(N)}{2}$

Problems:

1 Find the number of Integers less than 600 and Prime to it. Also find its Sums.

$$N = p^a q^b r^c$$

$$\phi(N) = N\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\left(1-\frac{1}{r}\right)$$

$$\begin{array}{r|l} 2 & 600 \\ \hline 2 & 300 \\ \hline 2 & 150 \\ \hline 3 & 75 \\ \hline 5 & 25 \\ \hline & 5 \end{array}$$

$$600 = 2^3 \times 3^1 \times 5^2$$

$$\phi(N) = 600\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$= 600 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5}$$

$$\phi(N) = 160$$

Sum of integers less than 600 and prime to it

$$\frac{N\phi(N)}{2}$$

$$= \frac{600 \times 160}{2}$$

$$= 48000$$

Result:

The highest power of Prime P contained in n factorial is 0 (or) $n/p + n/p^2 + \dots + \frac{n}{pk-1}$ where $n/pk = 0$ according as $n < P$ (or) $n \geq P$

2. Find the highest power of 2 in 10!

$$\left[\frac{10}{2}\right] = 5$$

$$\left[\frac{10}{2^2}\right] = \left[\frac{5}{2}\right] = 2$$

$$\left[\frac{10}{2^3}\right] = \left[\frac{2}{2}\right] = 1.$$

$$\left[\frac{10}{2^4}\right] = \left[\frac{1}{2}\right] = 0$$

$\therefore$ The highest power of 2 in 10!

$= 5+2+1$

$= 8$

3 Find the highest power of 3 in 1000!

$$\left[\frac{1000}{3}\right] = 333$$

$$\left[\frac{1000}{3^2}\right] = \left[\frac{333}{3}\right] = 111$$

$$\left[\frac{1000}{3^3}\right] = \left[\frac{111}{3}\right] = 37$$

$$\left[\frac{1000}{3^4}\right] = \left[\frac{37}{3}\right] = 12$$

$$\left[\frac{1000}{3^5}\right] = \left[\frac{12}{3}\right] = 4$$

$$\left[\frac{1000}{3^6}\right] = \left[\frac{4}{3}\right] = 1$$

$$\left[\frac{1000}{3^7}\right] = \left[\frac{1}{3}\right] = 0$$

The highest power of 3 in 1000!

$= 333 + 111 + 37 + 12 + 4 + 1$

$= 498$

4. Find the number of zeros with 61! ends.

Soln

Let us first find the highest power of 2 and 5 in 61!

$$\left[\frac{61}{2}\right] = 30$$

$$\left[\frac{61}{2^2}\right] = \left[\frac{30}{2}\right] = 15$$

$$\left[\frac{61}{2^3}\right] = \left[\frac{15}{2}\right] = 7$$

$$\left[\frac{61}{2^4}\right] = \left[\frac{7}{2}\right] = 3$$

$$\left[\frac{61}{2^5}\right] = \left[\frac{3}{2}\right] = 1$$

$$\left[\frac{61}{2^6}\right] = \left[\frac{1}{2}\right] = 0$$

The highest power of 2 in 61! = 30+15+7+3+1 = 56

The highest power of 5 in 61!

$$\left[\frac{61}{5}\right] = 12$$

$$\left[\frac{61}{5^2}\right] = \left[\frac{12}{5}\right] = 2$$

$$\left[\frac{61}{5^3}\right] = \left[\frac{2}{5}\right] = 0$$

The highest power of 5 in 61! = 12+2+0 = 14

∴ 61! Ends with 14 zeros min (14, 56)

5 Find the number of divisors and the sum of divisor of 480

$$N = p^a q^b r^c$$

Number of divisors $= (a+1)(b+1)(c+1)$

Sum of divisors $= \dfrac{p^{a+1}-1}{p-1} \quad \dfrac{q^{b+1}-1}{q-1} \quad \dfrac{r^{c+1}-1}{r-1}$

$$480 = 2^5 \times 3^1 \times 5^1$$

$$\begin{array}{r|l} 2 & 480 \\ \hline 2 & 240 \\ \hline 2 & 120 \\ \hline 2 & 60 \\ \hline 2 & 30 \\ \hline 3 & 15 \\ \hline & 5 \end{array}$$

No of divisors : $(5+1)(1+1)(1+1)$

$= 6 \times 2 \times 2$

$= 24$

Sum of divisors

$p=2,\ q=3,\ r=5,\ a=5,\ b=1,\ c=1$

$$= \frac{2^6-1}{2-1} \cdot \frac{3^2-1}{3-1} \cdot \frac{5^2-1}{5-1}$$

$$= \frac{64-1}{1} \cdot \frac{9-1}{2} \cdot \frac{25-1}{4}$$

$$= 63 \times \frac{\cancel{8}}{\cancel{2}} \times \frac{24}{\cancel{4}}$$

$$= 63 \times 24$$

$$= 1512$$

<u>Amicable Numbers:</u>

The numbers are said to be amicable numbers if the sum of the divisors excluding the number is equal to the other number.

6 Verify that 220 and 284 are amicable numbers

$$N = p^a q^b r^c$$

Sum of divisors $= \frac{p^{a+1}-1}{p-1} \cdot \frac{q^{b+1}-1}{q-1} \cdot \frac{r^{c+1}-1}{r-1}$

$$\begin{array}{r|l} 2 & 220 \\ \hline 2 & 110 \\ \hline 5 & 55 \\ \hline & 11 \end{array}$$

$$220 = 2^2 \times 5^1 \times 11^1$$

$p = 2,\ q = 5,\ r = 11$

$a = 2,\ b = 1,\ c = 1$

Sum of divisors $= \frac{2^{2+1}-1}{2-1} \cdot \frac{5^{1+1}-1}{5-1} \cdot \frac{11^{1+1}-1}{11-1}$

$= \frac{7}{1} \times \frac{24}{4} \times \frac{120}{10}$

$= 504$

Sum of divisors excluding the number $= 504 - 220 = 284$

$$\begin{array}{r|l} 2 & 284 \\ \hline 2 & 142 \\ \hline & 71 \end{array}$$

$$284 = 2^2 \times 71^1$$

$p = 2,\ q = 71$

$a = 2 \quad b = 1$

Sum of divisors $= \frac{2^{2+1}-1}{2-1} \cdot \frac{71^{1+1}-1}{71-1}$

$= \frac{7}{1} \times \frac{5040}{70}$

$= 504$

Sum of divisor excluding the number is

$= 504 - 284$

$= 220$

$\therefore$ 220 and 284 are amicable numbers.

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7. Find the smallest number with 30 divisors.

$$N = p^a q^b r^c \dots$$

Number of divisors $N = (a+1)(b+1)(c+1)$

$$N = 30 = 2^1 \times 3^1 \times 5^1$$

$$\begin{array}{r|l} 2 & 30 \\ \hline 3 & 15 \\ \hline & 5 \end{array}$$

$$30 = (a+1)(b+1)(c+1)$$

$$(a+1) = 5 \Rightarrow a = 4$$

$$(b+1) = 3 \Rightarrow b = 2$$

$$(c+1) = 2 \Rightarrow c = 1$$

$$N = 2^4 \times 3^2 \times 5^1$$

$$= 16 \times 9 \times 5$$

$$= 720$$

Congruences:

Two numbers a and b are said to be congruent with respect to modulo m. If they leave the same remainder when divided by m. We denote this by $a \equiv b \pmod m$

Note:

If $a \equiv b \pmod m$ implies $(a-b)$ is divisible by m

If $a \equiv b \pmod m$ then we can write $a-b = km$ where k is an integer.

Fermat's Theorem

Statement:

If P is a prime and a is any number prime to p then $a^{P-1}-1$ is divisible by P.

Proof:

Let $f(a) = a^P - a$

$$f(1) = 1^P - 1 = 0 \pmod P$$

$\therefore$ The result is true for $n=1$

Assume that the result is true for $a=n$ where n is a given integer

$$f(n) = n^P - n \pmod P$$

We will prove that the result is true for $a=n+1$ that is to prove that

$$f(n+1) = (n+1)^P - (n+1) = 0 \pmod P$$

$$f(n+1) = (n+1)^P - (n+1)$$

$$= n^P + PC_1 n^{P-1} + PC_2 n^{P-2} + \dots + PC_{P-1} n + 1 - (n+1)$$

$$= (PC_1 n^{P-1} + PC_2 n^{P-2} + \dots + PC_{P-1} n) + (n^P - n)$$

$$= (n^P - n) \bmod P$$

Since $PC_1, PC_2 \dots$ are divisible by P

$$= 0 \pmod P \text{ if } f(n) \text{ is true}$$

$$\therefore f(n+1) = 0 \pmod P$$

$$n^{12}-1 \equiv 0 \pmod{13}$$

also $n(n-1)$ is divisible by 2 being two consecutive integers.

All these being factors of $n^{13}-n$ we have

$$n^{13}-n \equiv 0 \mod (2,3,5,7,13).$$

$$n^{13}-n \equiv 0 \pmod{2730}$$

Hence Proved.

9 Prove that any square number is of the form $5n$ or $5n \pm 1$

soln:

If N is not prime to 5 then N is of the form $5n$ where n is some positive integer.

If N is prime to 5 then by Fermat's theorem N^4-1 is a multiple of 5

$$N^4-1 = 5n$$

$$((N^2)^2-(1)^2) = 5n$$

$$(N^2+1)(N^2-1) = 5n$$

$$N^2+1=5n \text{ or } N^2-1=5n$$

$$N^2 = 5n \pm 1$$

Wilson's Theorem

If P is a Prime number then $\lfloor P-1 + 1$ is divisible by P ie $\lfloor P-1 + 1 \equiv 0 \pmod{P}$

This result is true for $a = n+1$

Since $f(1)$ is true, $f(2), f(3), \ldots$ are all true

$\therefore f(n) = 0 \pmod{p}$ is true for all positive values of n.

$\therefore a^{p-1} - 1$ is divisible by p

Hence Proved.

8. Show that $n^{13} - n$ is divisible by 2730 if n is Prime to 2730

$$\begin{array}{r|l} 2 & 2730 \\ \hline 3 & 1365 \\ \hline 5 & 455 \\ \hline 7 & 91 \\ \hline & 13 \end{array}$$

$2730 = 2 \times 3 \times 5 \times 7 \times 13$

n is Prime to $2, 3, 5, 7, 13$

$$n^{13} - n = n(n^{12} - 1)$$

$$= n(n^6 - 1)(n^6 + 1)$$

$$= n(n^3 - 1)(n^3 + 1)(n^6 + 1)$$

$$= n(n^2 - 1)(n^2 + n + 1)(n^2 - n + 1)(n^6 + 1)$$

$$= n(n-1)(n^2 + n + 1)(n+1)(n^2 - n + 1)(n^6 + 1)$$

$$= (n^2 - 1)(n^4 + n^2 + 1)(n^6 + 1)$$

$$n^{13} - n = n(n^4 - 1)(n^8 + n^4 + 1)$$

Since n is Prime to $2, 3, 5, 7, 13$

$$n^2 - 1 \equiv 0 \pmod{3}$$

$$n^4 - 1 \equiv 0 \pmod{5}$$

$$n^6 - 1 \equiv 0 \pmod{7}$$

10 Prove that $\lfloor\underline{18} + 1$ is divisible by 437

$$437 = 19 \times 23$$

19 and 23 are Primes

By Wilson's Theorem

$$\lfloor\underline{19-1} + 1 \equiv 0 \pmod{19}$$

$$\lfloor\underline{18} + 1 \equiv 0 \pmod{19} \rightarrow (1)$$

$$\lfloor\underline{23-1} + 1 \equiv 0 \pmod{23}$$

$$\lfloor\underline{22} + 1 \equiv 0 \pmod{23}$$

$$22 \cdot 21 \cdot 20 \cdot 19 \lfloor\underline{18} + 1 \equiv 0 \pmod{23}$$

$$(23-1)(23-2)(23-3)(23-4)\lfloor\underline{18} + 1 \equiv 0 \pmod{23}$$

$$(-1)(-2)(-3)(-4)\lfloor\underline{18} + 1 \equiv 0 \pmod{23}$$

$$24 \lfloor\underline{18} + 1 \equiv 0 \pmod{23}$$

$$(23+1)\lfloor\underline{18} + 1 \equiv 0 \pmod{23}$$

$$(1)\lfloor\underline{18} + 1 \equiv 0 \pmod{23}$$

$$\lfloor\underline{18} + 1 \equiv 0 \pmod{23} \rightarrow (2)$$

Hence from (1) and (2)

$$\lfloor\underline{18} + 1 \equiv 0 \bmod (19, 23)$$

$\therefore$ $\lfloor\underline{18} + 1$ is divisible by 437.

11 If A and B are Prime to 1365 then show that

$a^{12} - b^{12} \equiv 0 \pmod{1365}$

$$1365 = 3 \times 5 \times 7 \times 13$$

$$\begin{array}{r|l} 3 & 1365 \\ \hline 5 & 455 \\ \hline 7 & 91 \\ \hline & 13 \end{array}$$

Since A and B are Prime to 1365

A and B are also prime to each of 3, 5, 7, 13

$$\therefore\ a^{12} - 1 \equiv 0 \pmod{13}$$

$$a^{6} - 1 \equiv 0 \pmod{7}$$

$$a^{4} - 1 \equiv 0 \pmod{5}$$

$$a^{2} - 1 \equiv 0 \pmod{3}$$

$$(a^{13} - a) = a(a^{12} - 1)$$

$$= a(a^{6} - 1)(a^{6} + 1)$$

$$= a(a^{3} - 1)(a^{3} + 1)(a^{6} + 1)$$

$$= a(a - 1)(a^{2} + a + 1)(a + 1)(a^{2} - a + 1)(a^{6} + 1)$$

$$= a(a^{2} - 1)(a^{4} + a^{2} + 1)(a^{6} + 1)$$

$$= a(a^{4} - 1)(a^{8} + a^{4} + 1)$$

$a^{2} - 1$, $a^{4} - 1$, $a^{6} - 1$, $a^{12} - 1$ are all factors of $a^{12} - 1$

$$\therefore\ a^{12} - 1 \equiv 0 \bmod (3, 5, 7, 13)$$

$$a^{12} - 1 \equiv 0 \bmod (1365) \rightarrow ①$$

$$\text{|||}^{\text{ly}}\quad b^{12} - 1 \equiv 0 \bmod (1365) \rightarrow ②$$

① − ②

$$\Rightarrow\ a^{12} - b^{12} \equiv 0 \pmod{1365}$$

12 Show that $16^{99} - 1 \equiv 0 \pmod{437}$

$437 = 19 \times 23$

$16^{99} - 1 = (2^4)^{99} - 1$

$= 2^{396} - 1$

$\phi(N) = N(1 - 1/p)(1 - 1/q)$

$= 437 \times \left(1 - 1/19\right)\left(1 - 1/23\right)$

$= 19 \times 23 \times \frac{18}{19} \times \frac{22}{23}$

$= 18 \times 22$

$= 396.$

By generalisation of Euler's theorem

$2^{396} - 1 \equiv 2^{\phi(437)} - 1 \equiv 0$

$\therefore 16^{99} - 1 \equiv 0 \pmod{437}$